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Trigonometry of ‘complex Hermitian’-type homogeneous symmetric spaces

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Abstract

This paper contains a thorough study of the trigonometry of the homogeneous symmetric spaces in the Cayley–Klein–Dickson family of spaces of ‘complex Hermitian’-type and rank-1. The complex Hermitian elliptic $\mathbb{C}P^N \equiv SU(N+1)/(U(1) \otimes SU(N))$ and hyperbolic $\mathbb{C}H^N \equiv SU(N,1)/(U(1) \otimes SU(N))$ spaces, their analogues with indefinite Hermitian metric $SU(p+1,q)/(U(1) \otimes SU(p,q))$ and the non-compact symmetric space $SL(N+1,\mathbb{R})/(SO(1,1) \otimes SL(N,\mathbb{R}))$ are the generic members in this family; the remaining spaces are some contractions of the former.

The method encapsulates trigonometry for this whole family of spaces into a single *basic trigonometric group equation*, and has ‘universality’ and ‘self-duality’ as its distinctive traits. All previously known results on the trigonometry of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ follow as particular cases of our general equations.

The following topics are covered rather explicitly: (0) description of the complete Cayley–Klein–Dickson family of rank-1 spaces of ‘complex type’, (1) derivation of the single basic group trigonometric equation, (2) translation to the basic ‘complex Hermitian’ cosine, sine and dual cosine laws, (3) comprehensive exploration of the bestiarium of ‘complex Hermitian’ trigonometric equations, (4) uncovering of a ‘Cartan’ sector of Hermitian trigonometry, related to triangle symplectic area and co-area, (5) existence conditions for a triangle in these spaces as inequalities and (6) restriction to the two special cases of ‘complex’ collinear and purely real triangles.

The physical quantum space of states of any quantum system belongs, as the complex Hermitian space member, to this parametrized family; hence its trigonometry appears as a rather particular case of the equations we obtain.

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1. Introduction

In a previous paper [1] the trigonometry of the complete family of symmetric spaces of *rank-1 and real type* was studied. These are also called Cayley–Klein (hereafter CK) real spaces which were first discussed by Klein, extending the Cayley idea of 'projective metrics'. In two dimensions, there are *nine* real spaces with a real quadratic symmetric metric of any constant curvature, and any metric (even degenerate) signature [2]. Further, the paper [1] had a long run aim towards opening an avenue for exploring the trigonometry of general symmetric homogeneous spaces.

Next to the spaces of real type, there are spaces of 'complex' type, coordinatized by elements of a one-step extension $\mathbb{R} \to {}_\eta\mathbb{R}$ through a labelled Cayley–Dickson procedure which adjoins an imaginary unit i with $\mathrm{i}^2 = -\eta$ to $\mathbb{R}$, producing either the complex, dual or split complex numbers according to $\eta > 0, = 0, < 0$. In 'complex' dimension N there are 3^{N+1} such geometries [3]; we will term these spaces Cayley–Klein–Dickson (CKD). They are 'Hermitian', since they are related to a scalar product with 'complex' values and Hermitian-like symmetry.

Within this family, spaces coordinatized by ordinary complex numbers (where $\eta > 0$ can be rescaled to $\eta = 1$ and $\mathrm{i}^2 = -1$) are actually *complex* spaces. There are 3^N CK complex-type geometries in complex dimension N [4]. For $N = 2$ we get nine 2D complex Hermitian CK spaces, with constant holomorphic curvature (either sign) and a Hermitian metric of any metric signature. All these are Hermitian symmetric spaces with a complex structure, hence Kählerian, but only the three spaces with a definite positive Hermitian metric belong to the restricted family of the so-called *two-point homogeneous spaces* [5]. These are the *elliptic Hermitian space*, i.e. the complex projective space $\mathbb{C}P^2$ with the Fubini–Study metric, the *Hermitian hyperbolic space* $\mathbb{C}H^2$ which can be realized in the interior of a Hermitian quadric in $\mathbb{C}P^2$ as its real analogue, and the Hermitian Euclidean space $\mathbb{C}R^2$, a two-dimensional (2D) Hilbert space, as the common 'limiting' space.

The full CK family of nine complex 2D spaces includes these three spaces with definite positive Hermitian metric, the complex anti-Newton–Hooke, Galilean and Newton–Hooke Hermitian planes (degenerate metric) and finally the anti-de Sitter, Minkowskian and de Sitter Hermitian planes (indefinite metric). The last six spaces are the complex Hermitian analogues of non-relativistic and relativistic spacetimes. In group theoretical terms, all these nine Hermitian spaces appear as *four* generic cases ($SU(3)/(U(1) \otimes SU(2))$, $SU(2,1)/(U(1) \otimes SU(2))$, $SU(2,1)/(U(1) \otimes SU(1,1))$, the last hosting two different spaces with 'time-like' and 'space-like' complex lines interchanged) and *five* non-generic ones, which are contractions of the four generic ones, with either curvature vanishing, metric degenerating or both. Within the full complex CK family of spaces of complex type, results on trigonometry are only available, as far as we know, for the three 2D Hermitian spaces with definite positive metric and constant (either positive or negative) holomorphic curvature [6–14]; a review of these results is included in section 2.

The nine spaces with $\eta < 0$ are much less known; they do not have a complex structure, but its 'split complex' analogue. Its generic members correspond to the symmetric homogeneous space $SL(3,\mathbb{R})/(SO(1,1) \otimes SL(2,\mathbb{R}))$, and the non-generic ones to some of its contractions. The nine spaces with $\eta = 0$ appear as common contractions from the spaces with $\eta > 0$ and $\eta < 0$. The trigonometry for such spaces has apparently not been studied.

In this paper, we set out the task of studying in full detail the *trigonometry of the complete family of spaces of 'complex' type*. It clearly suffices to consider the 2D case, as a triangle in any CKD such 'complex'-type space in 'complex' dimension N is fully contained in a totally geodesic subspace with 'complex' dimension 2.

The approach has several distinctive traits. First, it covers at once the trigonometry in the whole family of $3^3 = 27$ such geometries, parametrized by three real labels $\eta; \kappa_1, \kappa_2$. The study of the trigonometry in the family as a whole is in fact easier than its study on a case-by-case basis. Second, it gives a clear view of several *duality* relationships between the trigonometry of these different spaces, and explicitly displays the self-duality of $\mathbb{C}P^2$ (analogously to S^2) that is completely hidden in the trigonometric equations derived in [9–11, 13]. Third, it affords new results: *all* previously known equations appear and many new ones are obtained. Fourth, the (contracted) non-generic cases which correspond to curvature vanishing ($\kappa_1 = 0$) and/or ('Fubini–Study') metric degenerating ($\kappa_2 = 0$ or $\eta = 0$) are dealt with at the same level unlike the generic ones, and provide a very interesting new result on the relation of triangle symplectic area with angular and lateral 'phase excesses'. And fifth, the presence of the additional Cayley–Dickson-type label η makes possible the consideration of a new type of contractions, encompassed by the limit $\eta \to 0$ whose physical meaning is worth exploring.

The paper is self-contained, but reference to [1] may be helpful, specially for motivations and general background. A condensed review of already known results on the trigonometry of $\mathbb{C}P^2$ and $\mathbb{C}H^2$ [7–13] is given in section 2. Here we rewrite these known equations in terms of the invariants we propose. Information on CKD spaces of 'complex' type is given in sections 3 and 4. Section 3 deals with the ordinary complex case, and therefore refers to a much more familiar situation. Section 4 describes the main new traits appearing when complex numbers are replaced by their 'complex' parabolic and split (hyperbolic) versions.

In section 5 the approach to the trigonometry proposed in [1] is developed in depth for the *complete* CKD family. The whole of trigonometry for all these spaces is encapsulated in a single *basic trigonometric group equation*, which involves sides, angles, *lateral phases* and *angular phases* and exhibits explicitly self-duality in the whole family. The clue is a choice of triangle invariants as the canonical parameters of two pairs of commuting isometries; a choice which should call the attention of any physicist educated in quantum mechanics. Dealing with many spaces at once, this equation gives a perspective on some relationships going far beyond any treatment devoted to the study of a single space. The behaviour of trigonometry when either the curvature vanishes or the metric degenerates is explicitly described through the CKD constants $\eta; \kappa_1, \kappa_2$. Duality is the main structural backbone in our approach, and the requirement to explicitly maintain duality in all expressions and at all stages acts as a kind of method 'fingerprint'. Cartan duality for symmetric spaces [15] appears here as the change of sign in either η, κ_1 or κ_2.

By writing the *basic trigonometric group equation* in the fundamental 'complex' representation of the (η, κ_1, κ_2-dependent) group of motions, a set of nine 'complex' equations follows. With these as starting point, we will explore in section 6 the rather unknown territory of 'complex Hermitian' trigonometric equations. The real trigonometry background makes this exploration easier by deliberately pursuing the analogies, while at the same time the relevant differences stand out clearly. The most interesting new trait is the splitting of the equations into two 'sectors'. The first involves quantities linked to Cartan generators of the motion group, where two new triangle invariants appear in a rather natural way; they play a specially important role since they are proportional to the symplectic area and co-area of the triangle. For the *Hermitian elliptic space* $\mathbb{C}P^2$, these quantities were first found by Blasckhe and Terheggen [7, 8] but the Cartan sector appears here for the first time even in $\mathbb{C}P^N$ or $\mathbb{C}H^2$. The other 'sector' is the 'complex' analogue of the whole real trigonometry; it hosts the *family form* of *all* equations previously known for $\mathbb{C}P^N$ or $\mathbb{C}H^2$, together with a large number of new ones. In the $\mathbb{C}P^2$ case ($\eta = 1; \kappa_1 = 1, \kappa_2 = 1$) all trigonometric functions of sides, angles, lateral and angular phases are the ordinary circular ones, and at a first look the whole paper can be read by restricting to this case.

The *basic trigonometric identity* for 'complex Hermitian' spaces is also directly linked to other product formulae which we believe are new. They can be considered as 'complex Hermitian' Gauss–Bonnet formulae, and contain 'complex Hermitian' trigonometry in a nutshell, being equivalent to the *basic trigonometric identity*. The subject of such 'exponential product formulae' appears as a step in our derivation (section 5.1) but it can be developed further and affords a number of new identities which will be discussed elsewhere.

2. A review on Hermitian trigonometry of $\mathbb{C}P^2$ and $\mathbb{C}H^2$

The Hermitian elliptic space [16–18], i.e. $\mathbb{C}P^N$ endowed with the natural Fubini–Study metric induced by the *real part* of the Hermitian canonical flat product in $\mathbb{C}^{N+1}$, is a homogeneous *Hermitian symmetric space*. It has a natural complex structure, and the Fubini–Study (hereafter FS) metric is Kählerian and has constant holomorphic curvature; the Kähler form is proportional to the *imaginary part* of the Hermitian metric in $\mathbb{C}^{N+1}$ [19, 20]. We choose the scale in the metric so that the maximum distance in $\mathbb{C}P^N$ equals $\pi/2$, the total length of any (closed) geodesic equals π and the constant *holomorphic* curvature is $K_{\text{hol}} = 4$; the ordinary sectional curvature K of the FS metric in $\mathbb{C}P^N$ (as a Riemannian space of real dimension $2N$) is *not* constant, and lies in the interval $1 \leqslant K \leqslant 4$.

The homogeneous symmetric character of $\mathbb{C}P^N$ makes possible an explicit study of its trigonometry, albeit more complicated than for $\mathbb{R}P^N$, which essentially reduces to spherical trigonometry [21]. Most of the previous works on Hermitian trigonometry introduce *a single* real invariant for each side (which seems natural as $\mathbb{C}P^N$ is a rank-1 space), but *two* real invariants for each vertex, which also is natural due to the presence of *two* commuting factors in the isotropy subgroup of a point ($\mathbb{C}P^N$ can be described as $SU(N+1)/(U(1) \otimes SU(N))$. For *side invariants*, the canonical choice is the distances in the FS metric a (respectively b, c) between vertices BC (respectively CA, AB); to avoid degenerate cases all papers quoted before enforce the restrictions $a, b, c < \pi/2$. In both $\mathbb{C}P^N$ and the Hermitian hyperbolic space $\mathbb{C}H^N$ [14], identified with a bounded domain of $\mathbb{C}P^N$ with the hyperbolic FS metric, each point $[z]$ is a *ray* in the space $\mathbb{C}^{N+1}$ [11]. Even if these vectors are chosen normalized, $\langle z, z\rangle = 1$, every ray in $\mathbb{C}^{N+1}$ still contains infinitely many normalized vectors differing only by a phase factor. Let z^A, z^B, z^C denote arbitrarily chosen *normalized* position vectors in $\mathbb{C}^{N+1}$ (defined up to a phase factor) for the three vertices $A \equiv [z^A]$, $B \equiv [z^B]$, $C \equiv [z^C]$. The length a of the side $a \equiv BC$ can be obtained from the Hermitian product in the ambient linear space through $\cos a\, \mathrm{e}^{(\mathrm{i}\epsilon_a)} := \langle z^B, z^C\rangle$. The phase ϵ_a is *not* a triangle invariant, as the vectors representing the vertices can still be modified by any phase factors.

Vertex invariants are defined in terms of the tangent space, seen as a *real* vector space with *complex structure*. The tangent subspace to a complex (projective) line l at O is a real 2D subspace invariant under the complex structure and can be thus identified with a complex one-dimensionl (1D) subspace; it contains a one-parameter family of real 1D subspaces, corresponding to a one-parameter set of FS geodesics through O and contained in l. For *vertex invariants* associated with a pair of FS geodesics intersecting at O, several real quantities can be used: (1) the *Hermitian* angle between the sides seen as complex projective lines, denoted by X; (2) the *Euclidean* or FS angle Λ between u, v computed as usual in the Riemannian FS metric $g(\,,\,) := \mathrm{Re}\langle\,,\,\rangle$ in $\mathbb{C}P^N$ or $\mathbb{C}H^N$ [22]; (3) the FS angle Θ between $\mathrm{i}u, v$:

$$X := \arccos\left(\frac{|\langle u, v\rangle|}{\|u\|\cdot\|v\|}\right) \qquad \Lambda := \arccos\left(\frac{\mathrm{Re}\langle u, v\rangle}{\|u\|\cdot\|v\|}\right) \qquad \Theta := \arccos\left(\frac{\mathrm{Im}\langle u, v\rangle}{\|u\|\cdot\|v\|}\right). \tag{2.1}$$

In addition to these, yet not independent, one can also consider (4) the holomorphy 'inclination' Υ of the real 2-flat spanned by u, v, also called Kähler angle, inclination angle, holomorphy angle, slant angle etc between u and v [23]; as this depends only on the real 2-flat spanned by u, v, and not on u, v separately, the names *holomorphy inclination* or *Kähler inclination* seem more appropriate. In terms of vectors u, t which span the given 2-plane and are furthermore FS orthogonal, the *holomorphy inclination* Υ is given by

$$\Upsilon = \arccos\left(\frac{\mathrm{Re}\langle \mathrm{i}u, t\rangle}{\|u\|\cdot\|t\|}\right) \qquad \text{where} \quad t = \alpha u + \beta v \quad \alpha, \beta \in \mathbb{R} \quad \mathrm{Re}\langle u, t\rangle = 0. \tag{2.2}$$

The holomorphy inclination measures how this real 2-flat separates from the unique real 2-flat $\mathbb{C}_u$ containing u and *invariant* under the complex structure. Finally, (5) another angular invariant Φ of the pair u, v, its *angular phase* (pseudoangle, Kasner angle [23]) is

$$\langle u, v\rangle = |\langle u, v\rangle|\, \mathrm{e}^{\mathrm{i}\Phi}. \tag{2.3}$$

This angle has not been explicitly used in previous works on trigonometry on $\mathbb{C}P^N$ or $\mathbb{C}H^N$; it is generically well defined for any two vectors in the tangent space at each point (i.e. between two intersecting FS geodesics with tangent vectors u, v at the intersection point) but becomes indeterminate when u, v are Hermitian orthogonal. The angular invariant Φ is obviously meaningless between *complex* lines in these spaces.

To sum up, there are several different choices available for two *independent* vertex invariants; see the review by Scharnhorst [23]. Authors studying trigonometry have made different choices and the following relations will be useful:

$$\cos\Lambda = \cos X\cos\Phi \qquad \cos\Upsilon\sin\Lambda = \cos X\sin\Phi \tag{2.4}$$

$$\sin X = \sin\Lambda\sin\Upsilon \qquad \sin\Psi = \sin\Lambda\cos\Upsilon \tag{2.5}$$

$$\cos^2 X = \cos^2\Lambda + \sin^2\Lambda\cos^2\Upsilon \qquad \sin^2\Lambda = \sin^2 X + \sin^2\Psi. \tag{2.6}$$

In choosing symbols for these angular invariants, we have tried to conform to the majority usage, but nevertheless we have systematically changed them to capital letters, which allows a clear and systematic typographic rendering of the self-duality of the equations we will propose, by means of upper/lower case letters.

2.1. Trigonometry in the Hermitian spaces $\mathbb{C}P^2$ and $\mathbb{C}H^2$

The oldest general result is the Coolidge (1921) *sine theorem* [6]: the sides a, b, c and angles A, B, C between the sides seen as complex lines are related by

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{2.7}$$

The papers by Blaschke and Terheggen (1939) [7, 8] (hereafter BT) contained the first *complete* approach to trigonometry in $\mathbb{C}P^2$. Unlike the phase ϵ_a of $\langle \boldsymbol{z}^B, \boldsymbol{z}^C\rangle$ which is meaningless as a quantity in $\mathbb{C}P^2$, the combination $\Omega := \epsilon_a + \epsilon_b + \epsilon_c$ is a triangle invariant, as can be clearly seen in the relation $\langle \boldsymbol{z}^A, \boldsymbol{z}^B\rangle\langle \boldsymbol{z}^B, \boldsymbol{z}^C\rangle\langle \boldsymbol{z}^C, \boldsymbol{z}^A\rangle = \cos a\cos b\cos c\,\mathrm{e}^{(\mathrm{i}\Omega)}$. For reasons explained below, we will change the BT notation ω to Ω. Let us now consider the (normalized) position vectors $\boldsymbol{Z}^a, \boldsymbol{Z}^b, \boldsymbol{Z}^c$ of the poles $[\boldsymbol{Z}^a], [\boldsymbol{Z}^b], [\boldsymbol{Z}^c]$ of the three sides a, b, c defined by BT in the ambient space $\mathbb{C}^3$ through a 'cross product' of the ambient vertex position vectors as $\overline{\boldsymbol{Z}^a} = \frac{\boldsymbol{z}^B\times\boldsymbol{z}^C}{\sin a}$, $\overline{\boldsymbol{Z}^b}$ and $\overline{\boldsymbol{Z}^c}$ being given by similar expressions with suitable cyclic permutations of sides and vertices (see [14] for information about Hermitian cross products in $\mathbb{C}^3$). Then the dual procedure produces three angles A, B, C between sides seen as complex lines and another quantity ω as $\cos A\,\mathrm{e}^{(\mathrm{i}\epsilon_A)} := \langle \boldsymbol{Z}^b, \boldsymbol{Z}^c\rangle$ and $\omega := \epsilon_A + \epsilon_B + \epsilon_C$;

these four quantities are dual to a, b, c, Ω. BT gave a complete set of equations for the Hermitian elliptic space trigonometry: the Coolidge sine law (2.7) and two new equations, which we will call Blaschke–Terheggen cosine theorem for sides and angles; the need of *four* quantities (e.g. a, b, c, Ω) to determine a triangle up to isometry in the elliptic Hermitian space follows from these equations:

$$\cos^2 a = \frac{\cos^2 A + \cos^2 B \cos^2 C - 2\cos A \cos B \cos C \cos\omega}{\sin^2 B \sin^2 C} \tag{2.8}$$

$$\cos^2 A = \frac{\cos^2 a + \cos^2 b \cos^2 c - 2\cos a \cos b \cos c \cos\Omega}{\sin^2 b \sin^2 c}. \tag{2.9}$$

Another approach was put forward by Shirokov (published posthumously by Rosenfeld), with two angular invariants at each vertex: the Riemannian FS angle Λ_C between the two sides a, b as real 1D FS geodesics and the holomorphy inclination Υ_C of the 2-flat spanned at the vertex C by the real tangent vectors to the sides a, b. The equations are

$$\frac{\sin a}{\sin\Lambda_A \sin\Upsilon_A} = \frac{\sin b}{\sin\Lambda_B \sin\Upsilon_B} = \frac{\sin c}{\sin\Lambda_C \sin\Upsilon_C} \tag{2.10}$$

$$\frac{\sin 2a}{\sin\Lambda_A \cos\Upsilon_A} = \frac{\sin 2b}{\sin\Lambda_B \cos\Upsilon_B} = \frac{\sin 2c}{\sin\Lambda_C \cos\Upsilon_C} \tag{2.11}$$

$$\cos^2 a = (\cos b \cos c + \sin b \sin c \cos\Lambda_A)^2 + \sin^2 b \sin^2 c \cos^2\Upsilon_A \sin^2\Lambda_A \tag{2.12}$$

$$\cos 2a = \cos 2b \cos 2c + \sin 2b \sin 2c \cos\Lambda_A - 2\sin^2 b \sin^2 c \sin^2\Upsilon_A \sin^2\Lambda_A \tag{2.13}$$

as well as similar cosine and double cosine equations for the sides b, c. Among all these equations, only *five* are functionally independent (e.g. (2.12) and (2.13) are equivalent). The SR sine theorem (2.10) is equivalent to the Coolidge sine law as a consequence of (2.5).

In 1989 Wu-Yi Hsiang [10] gave a new derivation valid simultaneously for the trigonometry of the two-point homogeneous rank-1 spaces of real, complex, quaternionic and Cayley octonionic type, both elliptic and hyperbolic. At each vertex, say C, Hsiang uses the three invariants C, Λ_C, Θ_C linked by a relation (2.6) (recall $\Theta_C = \pi/2 - \Psi_C$). In the elliptic/hyperbolic case, he obtained some equations which (translated to C, Λ_C, Ψ_C) are given below (the first lines of equations (2.14) and (2.15)) as well as a rather complicated form of 'cosine theorem' that should not be considered as a 'basic' equation.

In 1990 Brehm [11] gave a fresh approach to trigonometry of $\mathbb{C}P^N$ and $\mathbb{C}H^N$. In terms of three angular invariants Λ_C, C, Ψ_C, only two of which are independent, Brehm obtains

$$\begin{gathered}\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \qquad \frac{\sin 2a}{\sin\Psi_A} = \frac{\sin 2b}{\sin\Psi_B} = \frac{\sin 2c}{\sin\Psi_C} \\ \cos 2a = \cos 2b \cos 2c + \sin 2b \sin 2c \cos\Lambda_A - 2\sin^2 b \sin^2 c \sin^2 A\end{gathered} \tag{2.14}$$

$$\begin{gathered}\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C} \qquad \frac{\sinh 2a}{\sin\Psi_A} = \frac{\sinh 2b}{\sin\Psi_B} = \frac{\sinh 2c}{\sin\Psi_C} \\ \cosh 2a = \cosh 2b \cosh 2c + \sinh 2b \sinh 2c \cos\Lambda_A - 2\sinh^2 b \sinh^2 c \sin^2 A.\end{gathered} \tag{2.15}$$

The first line equations are the Coolidge sine law and Hsiang form of double sine law. The last line turns out to be the Shirokov–Rosenfeld cosine double theorem for sides expressed in Brehm's angular variables. Brehm introduced the *triangle shape invariant*

$\sigma := \cos a \cos b \cos c \cos \Omega$, or $\sigma = -\cosh a \cosh b \cosh c \cos \Omega$, showed a triangle is completely determined up to isometry by a, b, c, σ (in the elliptic case Brehm assumes $a, b, c < \pi/2$), and gave inequalities that must be fulfilled for the triangle to exist as well as a careful discussion on congruence theorems.

In 1994 Hangan and Masala [24] gave an interpretation of Ω in $\mathbb{C}P^2$ as equal to twice the symplectic area enclosed by the triangle. The symplectic area comes from the Kähler structure of $\mathbb{C}P^2$, and is well defined by the triangle 'skeleton' itself, due to the closed nature of the Kähler form (the symplectic area of any surface depends only on the boundary).

The existence of two distinguished types of triangles is clear. In $\mathbb{C}P^2$, when $\Upsilon_A = 0$, then $\Upsilon_B = \Upsilon_C = 0$ follows and the trigonometry equations reduce to those of a spherical triangle in a sphere of curvature $K = 4$; this is seen in (2.11), (2.13) and corresponds to a triangle in a complex line $\mathbb{C}P^1$. When $\Upsilon_A = \pi/2$, then $\Upsilon_B = \Upsilon_C = \pi/2$ and the equations reduce (locally) to those of a spherical triangle in curvature $K = 1$; this can be seen in (2.10), (2.12) and corresponds to a triangle in a real projective subplane $\mathbb{R}P^2$, whose trigonometry comes from the spherical one after antipodal identification; in this case the values $e^{(i\Omega)}$ and $e^{(i\omega)}$ are *real*, as implied by the Blaschke–Terheggen equations (2.8) and (2.9). In these two special cases, the triangle is contained in a totally geodesic submanifold, whose sectional curvature attains the extremal values 4 and 1. In all other situations, the triangle is *not* contained in a totally geodesic submanifold. The sectional curvature of either $\mathbb{C}P^2$ or $\mathbb{C}H^2$ along any real 2-direction depends only on its holomorphy inclination Υ and is $K = \pm(4\cos^2\Upsilon + \sin^2\Upsilon)$.

3. The family of nine complex Hermitian Cayley–Klein 2D geometries and their spaces

3.1. The nine complex Hermitian Cayley–Klein 2D geometries

Let us consider a complex Hermitian form $(z, w) \to \langle z, w\rangle = \sum_{i,j}^{N} \overline{z_i}\Lambda_{ij}w_j$ in the complex linear space $\mathbb{C}^{N+1} = (z^0, z^1, \ldots, z^N)$ where the symmetric real matrix Λ is diagonal with entries $\{1, \kappa_1, \kappa_1\kappa_2, \ldots, \kappa_1\kappa_2\cdots\kappa_N\}$, depending on N real numbers κ_i. Linear isometries in $\mathbb{C}^{N+1}$ for such a Hermitian product close the special unitary CK group $SU_{\kappa_1,\kappa_2,\ldots,\kappa_N}(N+1)$ with Lie algebra $\mathfrak{su}_{\kappa_1,\kappa_2,\ldots,\kappa_N}(N+1)$. The structure of algebras in this family (any dimension) is described in [4].

When particularized for $N = 2$ we get a two-parametric family $SU_{\kappa_1,\kappa_2}(3)$ of groups: the eight-dimensional linear isometry groups of a complex Hermitian form $(z, w) \to \langle z, w\rangle = \sum_{i,j}^{3} \overline{z_i}\Lambda_{ij}w_j$ in a linear space $\mathbb{C}^3 = (z^0, z^1, z^2)$, with $\Lambda = \mathrm{diag}\{1, \kappa_1, \kappa_1\kappa_2\}$. When κ_1, κ_2 are different from zero, $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ is simple. This algebra is isomorphic to $\mathfrak{su}(3)$ when both κ_1, κ_2 are positive or to $\mathfrak{su}(2, 1)$ when at least one is negative. In the natural CK basis $\{P_1, P_2, Q_1, Q_2; J, M, B, I\}$ the CK algebra $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ has the following (fundamental or vectorial) three-dimensional (3D) complex matrix representation, where i is the complex unit:

$$P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 0 & \mathrm{i}\kappa_1 & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} 0 & 0 & \mathrm{i}\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\kappa_2 \\ 0 & \mathrm{i} & 0 \end{pmatrix} \tag{3.1}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix} \qquad I = \begin{pmatrix} \frac{-2\mathrm{i}}{3} & 0 & 0 \\ 0 & \frac{\mathrm{i}}{3} & 0 \\ 0 & 0 & \frac{\mathrm{i}}{3} \end{pmatrix}.$$

The generators B, I span a 2D Cartan subalgebra (regardless of the values of κ_i); further references to *the Cartan subalgebra* will mean this fiducial subalgebra. Let us introduce four new Cartan generators, given in the vector representation as

$$T_1 = \frac{1}{2}(I + B) = \begin{pmatrix} -\frac{\mathrm{i}}{3} & 0 & 0 \\ 0 & -\frac{\mathrm{i}}{3} & 0 \\ 0 & 0 & \frac{2\mathrm{i}}{3} \end{pmatrix} \qquad T_2 = \frac{1}{2}(I - B) = \begin{pmatrix} -\frac{\mathrm{i}}{3} & 0 & 0 \\ 0 & \frac{2\mathrm{i}}{3} & 0 \\ 0 & 0 & -\frac{\mathrm{i}}{3} \end{pmatrix}$$
$$H_1 = \frac{1}{2}(3I - B) = \begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \frac{1}{2}(3I + B) = \begin{pmatrix} -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}. \tag{3.2}$$

The Lie commutators of all these generators are given in (4.1) for $\eta = 1$. The CK algebras $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ can be endowed with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group of involutions generated by

$$\begin{aligned} \Pi_{(1)} &: (P_1, P_2, Q_1, Q_2; J, M, B, I) \to (-P_1, -P_2, -Q_1, -Q_2; J, M, B, I) \\ \Pi_{(2)} &: (P_1, P_2, Q_1, Q_2; J, M, B, I) \to (P_1, -P_2, Q_1, -Q_2; -J, -M, B, I). \end{aligned} \tag{3.3}$$

Each involution Π (either $\Pi_{(1)}, \Pi_{(2)}$ or $\Pi_{(02)} = \Pi_{(1)} \cdot \Pi_{(2)}$) determines a subalgebra $\mathfrak{h}$ of $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ invariant under Π; the subgroups generated by these subalgebras $\mathfrak{h}$ will be denoted by H, all with suitable subindices. The three Lie subalgebras $\mathfrak{h}_{(1)}, \mathfrak{h}_{(2)}$ and $\mathfrak{h}_{(02)}$ are of unitary CK type, $\mathfrak{u}_\kappa(2) \equiv \mathfrak{u}(1) \oplus \mathfrak{su}_\kappa(2)$ with $\kappa = \kappa_2, \kappa_1, \kappa_1\kappa_2$ respectively. Namely,

- the subalgebra $\mathfrak{h}_{(1)}$ is spanned by $I; J, M, B$ which close a $\mathfrak{u}_{\kappa_2}(2)$ (with I commuting with J, M, B). The group $H_{(1)}$ is isomorphic to $U(1) \otimes SU_{\kappa_2}(2)$.
- The subalgebra $\mathfrak{h}_{(2)}$ is spanned by $T_1; P_1, Q_1, H_1$ which close a $\mathfrak{u}_{\kappa_1}(2)$ (with T_1 commuting with P_1, Q_1, H_1). The group $H_{(2)}$ is isomorphic to $U(1) \otimes SU_{\kappa_1}(2)$.
- The subalgebra $\mathfrak{h}_{(02)}$ is spanned by $T_2; P_2, Q_2, H_2$ closing a $\mathfrak{u}_{\kappa_1\kappa_2}(2)$ (with T_2 commuting with P_2, Q_2, H_2). The group $H_{(02)}$ is isomorphic to $U(1) \otimes SU_{\kappa_1\kappa_2}(2)$.

The $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ generators can be represented in a pictorial way in a block triangular diagram, where each 'block' involves the four generators of the $\mathfrak{u}_\kappa(2)$ subalgebras listed above,

$$\begin{array}{ll} P_1 Q_1 & P_2 Q_2 \\ T_1 H_1 & T_2 H_2 \\ & J M \\ & I B \end{array} \tag{3.4}$$

and the generator in the $\mathfrak{u}(1)$ central subalgebra inside each $\mathfrak{u}_\kappa(2)$ appears at the left-lower corner in each block. The global block pattern extends the analogous pattern made by the three generators P_1, P_2, J of a real-type CK algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ and will be extremely helpful for visualization of most properties discussed below. For instance, a good way of writing the quadratic Casimir in $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$, with each group of terms corresponding to one of the three $\mathfrak{su}_\kappa(2)$-like subalgebras is given in (4.3) with $\eta = 1$.

The elements defining a 2D CK complex Hermitian geometry are similar to those in the real case [1, 25]. A 2D *complex CK geometry* is a set of three symmetric homogeneous spaces of points (linked to the involution $\Pi_{(1)}$) and lines of first and second kind (linked to $\Pi_{(2)}$ and $\Pi_{(02)}$). Details are rather analogous to the real case discussed in [1], so we restrict here to considering the symmetric homogeneous space of points,

$$\mathbb{C}S^2_{[\kappa_1],\kappa_2} \equiv SU_{\kappa_1,\kappa_2}(3)/H_{(1)} \equiv SU_{\kappa_1,\kappa_2}(3)/(U(1) \otimes SU_{\kappa_2}(2)) \qquad H_{(1)} = \langle I; J, M, B\rangle \tag{3.5}$$

Table 1. The nine 2D complex Hermitian CK geometries. At each entry the group G and the three subgroups $H_{(1)}$, $H_{(2)}$, $H_{(02)}$ are displayed.

	Measure of distance		
Measure of angle	Elliptic $\kappa_1 = 1$	Parabolic $\kappa_1 = 0$	Hyperbolic $\kappa_1 = -1$
Elliptic $\kappa_2 = 1$	Hermitian elliptic $SU(3)$ $H_{(1)} = U(1) \otimes SU(2)$ $H_{(2)} = U(1) \otimes SU(2)$ $H_{(02)} = U(1) \otimes SU(2)$	Hermitian Euclidean $IU(2)$ $H_{(1)} = U(1) \otimes SU(2)$ $H_{(2)} = U(1) \otimes IU(1)$ $H_{(02)} = U(1) \otimes IU(1)$	Hermitian hyperbolic $SU(2, 1)$ $H_{(1)} = U(1) \otimes SU(2)$ $H_{(2)} = U(1) \otimes SU(1, 1)$ $H_{(02)} = U(1) \otimes SU(1, 1)$
Parabolic $\kappa_2 = 0$	Hermitian co-Euclidean Hermitian oscillating NH $IU(2)$ $H_{(1)} = U(1) \otimes IU(1)$ $H_{(2)} = U(1) \otimes SU(2)$ $H_{(02)} = U(1) \otimes IU(1)$	Hermitian Galilean $IIU(1)$ $H_{(1)} = U(1) \otimes IU(1)$ $H_{(2)} = U(1) \otimes IU(1)$ $H_{(02)} = U(1) \otimes IU(1)$	Hermitian co-Minkowskian Hermitian expanding NH $ISU(1, 1)$ $H_{(1)} = U(1) \otimes IU(1)$ $H_{(2)} = U(1) \otimes SU(1, 1)$ $H_{(02)} = U(1) \otimes IU(1)$
Hyperbolic $\kappa_2 = -1$	Hermitian co-hyperbolic Hermitian anti-de Sitter $SU(2, 1)$ $H_{(1)} = U(1) \otimes SU(1, 1)$ $H_{(2)} = U(1) \otimes SU(2)$ $H_{(02)} = U(1) \otimes SU(1, 1)$	Hermitian Minkowskian $IU(1, 1)$ $H_{(1)} = U(1) \otimes SU(1, 1)$ $H_{(2)} = U(1) \otimes IU(1)$ $H_{(02)} = U(1) \otimes IU(1)$	Hermitian doubly hyperbolic Hermitian de Sitter $SU(2, 1)$ $H_{(1)} = U(1) \otimes SU(1, 1)$ $H_{(2)} = U(1) \otimes SU(1, 1)$ $H_{(02)} = U(1) \otimes SU(2)$

whose dimension (over $\mathbb{C}$) is 2. The generators I and J, M, B leave a point O (the origin) invariant, and so generate a direct product $U(1) \otimes SU_{\kappa_2}(2)$ of 'rotations' about O. The involution $\Pi_{(1)}$ is the reflection around O and P_1, Q_1 (respectively P_2, Q_2) move O and generate translations along the (complex) *basic* direction l_1 (respectively l_2).

The space $\mathbb{C}S^2_{[\kappa_1],\kappa_2}$ has a *complex Hermitian metric* with an associated *real 'Fubini-Study' metric (FS)*, $g(\,,\,) := \mathrm{Re}\langle\,,\,\rangle$ i.e. the real part of the Hermitian product, of constant holomorphic curvature $4\kappa_1$. This metric can also be derived from the Casimir and at the origin O the Hermitian product is given by the matrix $\mathrm{diag}(1, \kappa_2)$, and the 'FS' metric by $\mathrm{diag}(1, 1, \kappa_2, \kappa_2)$ (basis ordering P_1, Q_1, P_2, Q_2); at other points they are uniquely determined by invariance. This 'FS' metric is definite positive when $\kappa_2 > 0$, degenerate for $\kappa_2 = 0$ and indefinite of real type $(2, 2)$ for $\kappa_2 < 0$; when $\kappa_2 = 1$ it is the ordinary FS metric (elliptic or hyperbolic). The line l_1 (respectively l_2) contains two 'FS' orthogonal geodesics through O, the orbits of O by the one-parameter subgroups generated by P_1 and Q_1 (respectively P_2 and Q_2). A suitable rescaling of generators P_1, J_{12} allows κ_1 (respectively κ_2) to be reduced to ± 1. Thus there are nine 2D complex Hermitian CK geometries with groups of motion and isotopy subgroups $H_{(1)}, H_{(2)}, H_{(02)}$ displayed in table 1.

A fundamental property of the whole scheme of CK geometries is the existence of an 'automorphism' of each family, called *ordinary duality* $\mathcal{D}$. It is well defined for any dimension, and for the 2D case it is given by the following family automorphism:

$$\mathcal{D}: \begin{cases} (P_1, Q_1, P_2, Q_2; J, M, H_2, T_2) \to (-J, -M, -P_2, -Q_2; -P_1, -Q_1, H_2, -T_2) \\ (\kappa_1, \kappa_2) \to (\kappa_2, \kappa_1). \end{cases} \tag{3.6}$$

Duality $\mathcal{D}$ leaves the general commutation rules ((4.1) for $\eta = 1$) invariant while it interchanges the corresponding constants $\kappa_1 \leftrightarrow \kappa_2$ and the involutions $\Pi_{(1)} \leftrightarrow \Pi_{(2)}$, and hence the space of

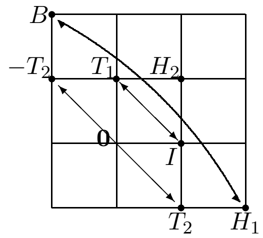

Figure 1. Fiducial Cartan subalgebra generators and their behaviour under duality.

points with the space of first-kind lines. It relates in general *two* different geometries placed in symmetrical positions relative to the main diagonal in table 1, just like in the real case. Duality also underlies the introduction of the Cartan generators (3.2): B, I form a natural basis for the fiducial Cartan subalgebra (B is the unique Cartan generator in the $SU_{\kappa_2}(2)$ part, and I in the $U(1)$ part, of the isotopy subalgebra of a point in $\mathbb{C}S^2_{[\kappa_1],\kappa_2}$), H_1, T_1 appear as their duals, and H_2, T_2 have the simplest behaviour under duality. In terms of the block-triangular arrangement (3.4), duality corresponds to a 'block reflection' along the secondary diagonal (and eventual sign change). For Cartan generators duality is depicted in figure 1. More details on the geometric interpretation of the Cartan subalgebra generators will be given later.

3.2. Realization of the spaces of points in the complex Hermitian Cayley–Klein spaces

Exponentiation of the matrix representation (3.1) and (3.2) of $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ produces a representation of $SU_{\kappa_1,\kappa_2}(3)$ as a linear transformations group in $\mathbb{C}^3 = (z^0, z^1, z^2)$. The one-parametric subgroups generated by P_1, P_2, Q_1, Q_2, J and M are

$$\begin{aligned}
\mathrm{e}^{P_1 x} &= \begin{pmatrix} C_{\kappa_1}(x) & -\kappa_1 S_{\kappa_1}(x) & 0 \\ S_{\kappa_1}(x) & C_{\kappa_1}(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} &\qquad \mathrm{e}^{P_2 x} &= \begin{pmatrix} C_{\kappa_1\kappa_2}(x) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(x) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(x) & 0 & C_{\kappa_1\kappa_2}(x) \end{pmatrix} \\
\mathrm{e}^{Q_1 x} &= \begin{pmatrix} C_{\kappa_1}(x) & \mathrm{i}\kappa_1 S_{\kappa_1}(x) & 0 \\ \mathrm{i}S_{\kappa_1}(x) & C_{\kappa_1}(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} &\qquad \mathrm{e}^{Q_2 x} &= \begin{pmatrix} C_{\kappa_1\kappa_2}(x) & 0 & \mathrm{i}\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(x) \\ 0 & 1 & 0 \\ \mathrm{i}S_{\kappa_1\kappa_2}(x) & 0 & C_{\kappa_1\kappa_2}(x) \end{pmatrix} \\
\mathrm{e}^{J x} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(x) & -\kappa_2 S_{\kappa_2}(x) \\ 0 & S_{\kappa_2}(x) & C_{\kappa_2}(x) \end{pmatrix} &\qquad \mathrm{e}^{M x} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(x) & \mathrm{i}\kappa_2 S_{\kappa_2}(x) \\ 0 & \mathrm{i}S_{\kappa_2}(x) & C_{\kappa_2}(x) \end{pmatrix}
\end{aligned} \tag{3.7}$$

where the cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ functions with label κ are defined by

$$C_\kappa(x) := \begin{cases} \cos\sqrt{\kappa}x \\ 1 \\ \cosh\sqrt{-\kappa}x \end{cases} \qquad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}x & \kappa < 0. \end{cases} \tag{3.8}$$

These functions coincide with the circular and hyperbolic trigonometric ones for $\kappa = 1$ and $\kappa = -1$; the case $\kappa = 0$ provides the so-called 'parabolic' or Galilean functions: $C_0(x) = 1$, $S_0(x) = x$. General properties of these functions are given in the appendix of [1].

The exponentials of the Cartan subalgebra generators B, I, T_1, T_2, H_1, H_2 are

$$
\begin{aligned}
&e^{Bx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-ix} & 0 \\ 0 & 0 & e^{ix} \end{pmatrix} \qquad
&& e^{Ix} = \begin{pmatrix} e^{\frac{-2ix}{3}} & 0 & 0 \\ 0 & e^{\frac{ix}{3}} & 0 \\ 0 & 0 & e^{\frac{ix}{3}} \end{pmatrix} \\
&e^{T_1x} = \begin{pmatrix} e^{\frac{-ix}{3}} & 0 & 0 \\ 0 & e^{\frac{-ix}{3}} & 0 \\ 0 & 0 & e^{\frac{2ix}{3}} \end{pmatrix} \qquad
&& e^{T_2x} = \begin{pmatrix} e^{\frac{-ix}{3}} & 0 & 0 \\ 0 & e^{\frac{2ix}{3}} & 0 \\ 0 & 0 & e^{\frac{-ix}{3}} \end{pmatrix} \\
&e^{H_1x} = \begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & e^{ix} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad
&& e^{H_2x} = \begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{ix} \end{pmatrix}.
\end{aligned}
\tag{3.9}
$$

Any element $U \in SU_{\kappa_1,\kappa_2}(3)$, satisfying $U^\dagger \Lambda U = \Lambda, \det U = 1$ can be written as a product of matrices (3.7), (3.9). The action of $SU_{\kappa_1,\kappa_2}(3)$ on $\mathbb{C}^3$ is linear but not transitive, since it conserves the Hermitian form $|z^0|^2 + \kappa_1|z^1|^2 + \kappa_1\kappa_2|z^2|^2$. The isotopy subgroup of $O = (1, 0, 0)$ is the three-parameter subgroup $SU_{\kappa_2}(2) = \langle J, M, B\rangle$, while the $U(1) = \langle I\rangle$ multiplies O by a phase factor. Hence the homogeneous symmetric space $\mathbb{C}S^2_{[\kappa_1],\kappa_2} \equiv SU_{\kappa_1,\kappa_2}(3)/(U(1) \otimes SU_{\kappa_2}(2))$ can be identified with the orbit of the *ray* $[O]$ under the action of $SU_{\kappa_1,\kappa_2}(3)$. This orbit is the domain $|z^0|^2 + \kappa_1|z^1|^2 + \kappa_1\kappa_2|z^2|^2 > 0$ of $\mathbb{C}P^2$, and when $\kappa_1 > 0, \kappa_2 > 0$ it is the full complex projective space $\mathbb{C}P^2$. The *Weierstrass coordinates* (z^0, z^1, z^2) are linked by $|z^0|^2 + \kappa_1|z^1|^2 + \kappa_1\kappa_2|z^2|^2 = 1$ and are still defined up to a common unimodular complex factor; these are the natural coordinates in the *vector models* of the Hermitian CK spaces, since the motion groups act linearly on them.

The non-generic situation where κ_1, κ_2 vanishes corresponds to an Inönü–Wigner contraction [26]. The limit $\kappa_1 \to 0$ is a local contraction (around a point); it carries the first and third columns of table 1 to the flat middle one. The limit $\kappa_2 \to 0$ is an axial contraction (around a line), carrying geometries of first and third rows to the middle one.

4. The complete family of 'complex Hermitian' Cayley–Klein–Dickson 2D geometries and their spaces

The previous section has been written so that it can be re-read with minimal *mutatis mutandis* changes to suit the description of the full family of 'complex-type' spaces. The new fact is the explicit Cayley–Dickson (CD) label $\eta = -i^2$ in the CD doubling $\mathbb{R} \to {}_\eta\mathbb{R}$: $x \to x + iy$. There are three cases: $\eta > 0$ which can be rescaled to $\eta = 1$ gives the division algebra of ordinary complex numbers ${}_{+1}\mathbb{R} \equiv \mathbb{C}$; $\eta = 0$ gives the *dual* or *Study* numbers ${}_0\mathbb{R} \equiv \mathbb{C}_0$; and $\eta < 0$ can be rescaled to $\eta = -1$ and gives the *split complex* numbers ${}_{-1}\mathbb{R} \equiv \mathbb{C}_{-1}$, also called *double*, hyperbolic complex, Lorentz or perplex numbers.

4.1. The 'complex Hermitian' Cayley–Klein–Dickson 2D geometries

The groups behind these geometries are the linear isometry groups of a 'complex Hermitian' form $(z, w) \to \langle z, w\rangle = \sum_{i,j} \overline{z_i}\Lambda_{ij} w_j$ in the 'complex' linear space $\mathbb{C}^{N+1}_\eta \equiv {}_\eta\mathbb{R}^{N+1} = (z^0, z^1, \ldots, z^N)$ with the same real Λ as in the complex case. For $N = 2$ the CKD algebra, denoted by ${}_\eta\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ is eight dimensional, and its fundamental or vectorial 3D 'complex' representation is given by 3×3 matrices (3.1), (3.2), where now entries are in $\mathbb{C}_\eta$ and i stands for the pure imaginary 'complex' unit in $\mathbb{C}_\eta$. This form has 'Hermitian' symmetry $\langle w, z\rangle = \overline{\langle z, w\rangle}$, with 'complex' conjugation in $\mathbb{C}_\eta$: $z = a + ib \to \overline{z} = a - ib$, for real

a, b and the form $(z, w) \to -\mathrm{Im}\langle z, w\rangle$ is still real and antisymmetric in z, w, and therefore is a symplectic form in the real space $\mathbb{R}^{2(N+1)}$ underlying ${}_\eta\mathbb{R}^{N+1}$ (the minus sign is just a convenience). To prevent ambiguity, we will keep the term *Hermitian* for the truly complex case, and we will put quotes in 'complex' and 'Hermitian' when referring to the general 'complex' numbers $\mathbb{C}_\eta$ with CD label η. A simple scale change may reduce simultaneously η and κ_1, κ_2 to $1, 0$ or -1. When $\eta = 1$, the CKD algebra ${}_+\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ is isomorphic to the CK Lie algebra $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$; it is simple when κ_1, κ_2 are different from zero. When $\eta = -1, \kappa_1 \neq 0, \kappa_2 \neq 0$, the CKD algebra ${}_-\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ is also simple and isomorphic to the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. Generators B, I, T_1, T_2, H_1, H_2 still belong to the fiducial 2D Cartan subalgebra of ${}_\eta\mathfrak{su}_{\kappa_1,\kappa_2}(3)$.

For any value of η the CKD algebras in the family ${}_\eta\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ can be endowed with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group of commuting involutive automorphisms generated by $\Pi_{(1)}, \Pi_{(2)}$ (3.3); denoting everything as in the previous section, the three Lie subalgebras $\mathfrak{h}_{(1)}$, $\mathfrak{h}_{(2)}$ and $\mathfrak{h}_{(02)}$ invariant under the involutions with the same indices are spanned by the generators with the same name as in the complex case and turn out to be of CKD type, ${}_\eta\mathfrak{u}(1) \oplus \mathfrak{su}_\kappa(2)$ with $\kappa = \kappa_2, \kappa_1, \kappa_1\kappa_2$ respectively. The groups they generate are isomorphic to ${}_\eta U(1) \otimes {}_\eta SU_{\kappa_2}(2)$, ${}_\eta U(1) \otimes {}_\eta SU_{\kappa_1}(2)$ and ${}_\eta U(1) \otimes {}_\eta SU_{\kappa_1\kappa_2}(2)$. In all these expressions, the Lie algebras of CKD 'unitary type' are ${}_\eta\mathfrak{u}_\kappa(2) \equiv {}_\eta\mathfrak{u}(1) \oplus {}_\eta\mathfrak{su}_\kappa(2)$, and for the groups of 'unimodular complex numbers' ${}_\eta U(1)$ we have two generic cases ${}_+U(1) \equiv U(1) \equiv SO(2)$, ${}_-U(1) \equiv SO(1, 1)$ and one limiting case ${}_0U(1) \equiv ISO(1) \equiv \mathbb{R}$.

The Lie algebra ${}_\eta\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ is given by the following Lie commutators:

$$
\begin{array}{llll}
[P_1, P_2] = \kappa_1 J & [P_2, Q_1] = \kappa_1 M & [Q_1, Q_2] = \eta\kappa_1 J & \\
[P_1, Q_1] = 2\kappa_1 H_1 & [P_2, Q_2] = 2\kappa_1\kappa_2 H_2 & & \\
[P_1, Q_2] = \kappa_1 M & & & \\
[P_1, J] = -P_2 & [P_2, J] = \kappa_2 P_1 & [Q_1, J] = -Q_2 & [Q_2, J] = \kappa_2 Q_1 \\
[P_1, M] = -Q_2 & [P_2, M] = -\kappa_2 Q_1 & [Q_1, M] = \eta P_2 & [Q_2, M] = \eta\kappa_2 P_1 \\
[P_1, B] = Q_1 & [P_2, B] = -Q_2 & [Q_1, B] = -\eta P_1 & [Q_2, B] = \eta P_2 \\
[P_1, I] = -Q_1 & [P_2, I] = -Q_2 & [Q_1, I] = \eta P_1 & [Q_2, I] = \eta P_2 \\
[P_1, T_1] = 0 & [P_2, T_1] = -Q_2 & [Q_1, T_1] = 0 & [Q_2, T_1] = \eta P_2 \\
[P_1, T_2] = -Q_1 & [P_2, T_2] = 0 & [Q_1, T_2] = \eta P_1 & [Q_2, T_2] = 0 \\
[P_1, H_1] = -2Q_1 & [P_2, H_1] = -Q_2 & [Q_1, H_1] = 2\eta P_1 & [Q_2, H_1] = \eta P_2 \\
[P_1, H_2] = -Q_1 & [P_2, H_2] = -2Q_2 & [Q_1, H_2] = \eta P_1 & [Q_2, H_2] = 2\eta P_2 \\
[J, M] = 2\kappa_2 B & & & \\
[J, B] = -2M & [J, I] = 0 & [M, B] = 2\eta J & [M, I] = 0 \\
[J, T_1] = -M & [J, T_2] = M & [M, T_1] = \eta J & [M, T_2] = -\eta J \\
[J, H_1] = M & [J, H_2] = -M & [M, H_1] = -\eta J & [M, H_2] = \eta J.
\end{array}
\tag{4.1}
$$

Now the *plane* as the set of points corresponds to the symmetric homogeneous space

$$
\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2} \equiv {}_\eta SU_{\kappa_1,\kappa_2}(3)/H_{(1)} \equiv {}_\eta SU_{\kappa_1,\kappa_2}(3)/({}_\eta U(1) \otimes {}_\eta SU_{\kappa_2}(2)) \qquad H_{(1)} = \langle I; J, M, B\rangle.
\tag{4.2}
$$

This space is again dimension 2 over $\mathbb{C}_\eta$, and all comments made in the complex case can be easily rephrased. The *ordinary duality* $\mathcal{D}$ (3.6) extends, with $\mathcal{D} : \eta \to \eta$, to an 'automorphism'

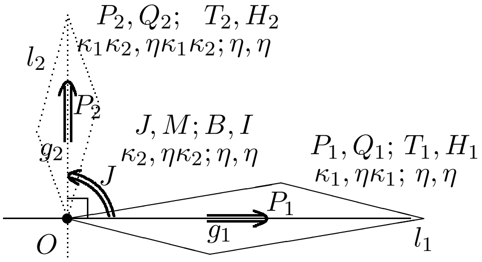

Figure 2. Generators and their associated labels in a 'complex Hermitian' 2D CKD geometry. Lines l_1, and l_2 are 'complex', thus two dimensional from a real point of view.

of the complete CKD family and leaves the commutation rules (4.1) invariant. In general $\mathcal{D}$ relates *two* different 'complex Hermitian' CKD geometries with the same label η but κ_1, κ_2 interchanged. Figure 2 displays the generators (with their labels) as related to the three fiducial elements O, l_1, l_2.

The quadratic Lie algebra Casimir in ${}_\eta\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ can be written grouping the terms which correspond to the three ${}_\eta\mathfrak{su}_\kappa(2)$-like subalgebras:

$$\mathcal{C} = \left(\left(\eta P_2^2 + Q_2^2\right) + \kappa_1\kappa_2 H_2^2\right) + \kappa_2\left(\left(\eta P_1^2 + Q_1^2\right) + \kappa_1 H_1^2\right) + \kappa_1((\eta J^2 + M^2) + \kappa_2 B^2). \tag{4.3}$$

From this Casimir, we can easily derive the invariant 'FS' metric in the space $\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2}$ which is given, at the origin and in the basis P_1, Q_1, P_2, Q_2 by the matrix $\mathrm{diag}(1, \eta, \kappa_2, \eta\kappa_2)$, coming also as the real part of the Hermitian product whose matrix at O is $\mathrm{diag}(1, \kappa_2)$.

4.2. Realization of spaces of points in the 'complex Hermitian' Cayley–Klein–Dickson spaces

When η is present, the fundamental 3D 'complex' matrix representation (3.1), (3.2) exponentiates to a representation of ${}_\eta SU_{\kappa_1,\kappa_2}(3)$ as a linear transformations group in $\mathbb{C}^3_\eta$. One-parametric subgroups are given again by (3.7), (3.9), where $\mathrm{e}^{\mathrm{i}x}$ is related to the sine and cosine with label η by a Euler-like formula:

$$\mathrm{e}^{\mathrm{i}x} = C_\eta(x) + \mathrm{i}S_\eta(x). \tag{4.4}$$

Again the action of ${}_\eta SU_{\kappa_1,\kappa_2}(3)$ on $\mathbb{C}^3_\eta$ is linear but not transitive, since it conserves the 'Hermitian' form $|z^0|^2 + \kappa_1|z^1|^2 + \kappa_1\kappa_2|z^2|^2$. The isotopy subgroup of the point $O \equiv (1, 0, 0)$ is easily seen to be the three parameter subgroup ${}_\eta SU_{\kappa_2}(2) = \langle J, M, B\rangle$, while the ${}_\eta U(1) = \langle I\rangle$ subgroup multiplies this vector by a unimodular 'complex' phase factor. Hence the homogeneous symmetric space $\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2} \equiv {}_\eta SU_{\kappa_1,\kappa_2}(3)/({}_\eta U(1) \otimes {}_\eta SU_{\kappa_2}(2))$ can be identified with the orbit of the *ray* $[O]$ under the action of the group ${}_\eta SU_{\kappa_1,\kappa_2}(3)$. The geometry behind the case $\eta < 0$ differs greatly from the ordinary complex case: for $\eta < 0, \kappa_2 \neq 0$ the FS metric is always indefinite and of $(2, 2)$ real type, irrespective of the sign of κ_2. The four spaces of points with $\eta < 0, \kappa_1 \neq 0, \kappa_2 \neq 0$ are essentially the same, though the choices of lines and FS geodesics of first and second kind are interchanged; this is tantamount to what happens in the real case for the (1 + 1)-anti-de Sitter and de Sitter spaces. These four spaces can be realized as spaces of 0-pairs in $\mathbb{R}P^2$ (pairs made from a point and a hyperplane (here line) in the *real* projective plane $\mathbb{R}P^2$), and the distance between two 0-pairs $(X; \alpha)$, $(Y; \beta)$ is related to the cross ratio of the four points $X, Y; Z, T$, where Z, T are the intersections of the line determined by X, Y with the hyperplanes α, β (see [13], theorems 2.39 and 4.21).

The non-generic situation where a coefficient $\eta; \kappa_1, \kappa_2$ vanishes corresponds to an Inönü–Wigner contraction [26]. The limit $\kappa_1 \to 0$ is a local contraction (around a point). The

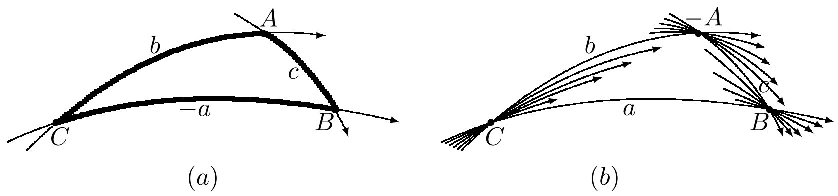

Figure 3. (*a*) Triangular point loop, (*b*) triangular line loop.

limit $\kappa_2 \to 0$ is a line contraction (around a whole 'complex' line). Finally, the limit $\eta \to 0$ corresponds to a new kind of contraction around a purely real submanifold, the projectivized real κ_1, κ_2 CK space.

5. The compatibility conditions for a triangular loop

In this section, we discuss the approach to the trigonometry of the 27 'complex Hermitian type' CKD spaces, and we introduce the 'complex Hermitian' compatibility equations, loop equations and the basic trigonometric identity. Some general comments on this approach are given in [1]; we emphasize the choice of 'external angles' at the vertex A and the fact that the standard angular excess appears without the explicit presence of the measure of twice a quadrant of angle (which equals π when $\kappa_2 = 1$).

A triangle in a 'complex Hermitian' CKD space can be seen either as a triangle point loop or dually as a triangle line loop (see figure 3). In the first case, a point C moves to a different point B 'translating' either along the geodesic segment CB or along the two geodesic segments CA and AB. Dually, the geodesic $c \equiv AB$ is considered to move to a different geodesic $b \equiv CA$ 'rotating' either about the vertex $A \equiv bc$ or about the vertices $B \equiv ca$ and then $C \equiv ab$. There is a very important difference with the real 2D case: the 'translations' along a geodesic are not uniquely defined by the geodesic only. Thus to make sense out of the idea of triangle loop, a closer analysis of the geometry is required. Any geodesic g through C determines a well-defined 'complex' line $\mathbb{C}_\eta g$ containing g. Thus for two geodesics a, b intersecting at C there are two uniquely determined 'complex' lines $\mathbb{C}_\eta a$, $\mathbb{C}_\eta b$ through C, which will lie on a (generically) well-defined *line-geodesic* G_C. This is dual to the determination of a (generically) well-defined geodesic g_a through two different points C, B. This subtlety is not necessary in the real case where a line contains a 1D set of points, while a complex line contains a 2D set of points.

To start with the study of trigonometry, we will take as 'side' and 'angle' measures the canonical parameters of certain one-parameter subgroup elements. To explain this choice, we first select a (real) flag $O \subset g_1 \subset l_1 \subset G_1$ as follows: O is the origin point $O = [(1, 0, 0)]$, g_1 is the orbit of O under the 1D subgroup generated by P_1 (thus the 'complex' line l_1 is the orbit of O under the subgroup generated by P_1, Q_1) and the line-geodesic G_1 is the orbit of l_1 under the subgroup generated by J; this flag is determined by isolating the generators P_1, Q_1, J. Now move the triangle to a canonical position where C coincides with O, the side a is on g_1 and the side b lies on a geodesic in G_1. In this 'canonical' position, the side b is obtained from a by means of two *commuting* rotations generated by J and by the *phase rotation generator* I, the *unique* generator in the fiducial Cartan subalgebra *commuting* with J. As angular invariants, we take the canonical parameters C ('Hermitian' or 'pure' angle between 'complex' lines) and Φ_C (*angular phase* between real 1D geodesics within a 'complex' line) of these two 'rotations' whose product, the *complete rotations* about C, brings side a to coincide with b:

$$\mathrm{e}^{CJ}\,\mathrm{e}^{\Phi_C I}. \tag{5.1}$$

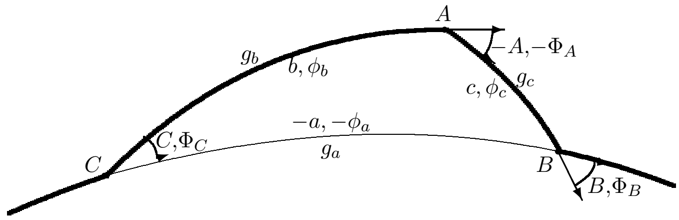

Figure 4. A 'complex Hermitian' triangular loop as a single curve.

Since 'Hermitian' CKD spaces are rank-1, each pair of points has a single invariant and it would seem enough to consider the FS distance a between the points C, B as the unique moduli of sides. This was done in previous works on Hermitian trigonometry [9–11, 13]. But the formal duality requirement prompts the consideration of translation partners to both $J; I$ and since duality maps $J; I$ onto $-P_1; -T_1$, this suggests the use of the *complete translations*:

$$\mathrm{e}^{aP_1}\,\mathrm{e}^{\phi_a T_1}. \tag{5.2}$$

There are geometrical reasons for the extra 'translation' $\mathrm{e}^{\phi_a T_1}$ in addition to the duality requirement: the 'pure' translation e^{aP_1} carries the vertex C to B, but this alone *does not* carry the unique 'complex line'-geodesic at the vertex C determined by $\mathbb{C}_\eta a, \mathbb{C}_\eta b$ to the 'complex' line-geodesic at B determined by $\mathbb{C}_\eta b, \mathbb{C}_\eta c$; an additional $\mathrm{e}^{\phi_a T_1}$ is needed. With 'complete' rotations and translations, duality is manifestly restored: at each vertex a complete rotation is required to bring into coincidence simultaneously the sides seen both as 'complex' lines and as point-geodesic sides. And along each side, a complete translation is required to bring into coincidence the vertices seen both as points and as 'complex' line-geodesics determined at each vertex by the two sides.

This choice of two *commuting* generators is very natural from a quantum mechanics viewpoint and affords six 'vertex' quantities (three Hermitian or pure angles A, B, C and three *angular phases* Φ_A, Φ_B, Φ_C) and six 'side' quantities (three lengths a, b, c and three *lateral phases* ϕ_a, ϕ_b, ϕ_c). All these invariants appear as canonical parameters of pairs of commuting isometries, generated respectively by J_A, J_B, J_C; I_A, I_B, I_C and P_a, P_b, P_c; T_a, T_b, T_c. At each side P_a, P_b, P_c are pure translation generators that perform the canonical parallel transport along their FS geodesic axes, and T_a, T_b, T_c are the only Cartan generators in the isotopy subalgebras of the sides a, b, c commuting with P_a, P_b, P_c.

The Cartan subalgebra is contained in the isotopy subalgebra of O, so its elements I, B, T_1, T_2, H_1, H_2 generate 'rotations' about O and the rotation $\mathrm{e}^{\Phi I}$ appears in the complete rotation around O. What about the phase 'rotation' part $\mathrm{e}^{\phi T_1}$ appearing as a part of the complete *translation* $\mathrm{e}^{xP_1}\,\mathrm{e}^{\phi_x T_1}$? Since any Cartan transformation as $\mathrm{e}^{\phi T_1}$ leaves pointwise invariant the 'complex' line l_1, it follows that T_1 should also be considered as generating 'translations' along l_1. The same happens for all other Cartan transformations which are somewhat 'hybrid'; they are *rotations* about a point and *translations* along a 'complex' line. This is suggested by diagram (3.4) as the *whole* Cartan subalgebra is contained in each of the three blocks, the isotopy subalgebras of l_1, l_2, O respectively.

From now on everything follows the real pattern [1], and the commutativity between both components of complete transformations allows the extension of the basic real identities to 'complex' ones: *compatibility* identities, *point loop and side loop* equations and *basic trigonometric identity*. The generators $P_a, T_a, P_b, T_b, P_c, T_c$; $J_A, I_A, J_B, I_B, J_C, I_C$ are not independent. They are related by several *compatibility conditions*, to be considered as an implicit group theoretical definition for the three sides, the three angles, the three lateral phases and the three angular phases:

$$\begin{pmatrix}P_b\\T_b\end{pmatrix} = e^{CJ_C}\, e^{\Phi_C I_C} \begin{pmatrix}P_a\\T_a\end{pmatrix} e^{-\Phi_C I_C}\, e^{-CJ_C} \qquad \begin{pmatrix}J_B\\I_B\end{pmatrix} = e^{cP_c}\, e^{\phi_c T_c} \begin{pmatrix}J_A\\I_A\end{pmatrix} e^{-\phi_c T_c}\, e^{-cP_c}$$
$$\begin{pmatrix}P_c\\T_c\end{pmatrix} = e^{-AJ_A}\, e^{-\Phi_A I_A} \begin{pmatrix}P_b\\T_b\end{pmatrix} e^{\Phi_A I_A}\, e^{AJ_A} \qquad \begin{pmatrix}J_C\\I_C\end{pmatrix} = e^{-aP_a}\, e^{-\phi_a T_a} \begin{pmatrix}J_B\\I_B\end{pmatrix} e^{\phi_a T_a}\, e^{aP_a} \tag{5.3}$$
$$\begin{pmatrix}P_a\\T_a\end{pmatrix} = e^{BJ_B}\, e^{\Phi_B I_B} \begin{pmatrix}P_c\\T_c\end{pmatrix} e^{-\Phi_B I_B}\, e^{-BJ_B} \qquad \begin{pmatrix}J_A\\I_A\end{pmatrix} = e^{bP_b}\, e^{\phi_b T_b} \begin{pmatrix}J_C\\I_C\end{pmatrix} e^{-\phi_b T_b}\, e^{-bP_b}.$$

All the trigonometry of the 'complex' CKD space is *completely* contained in these equations, which are explicitly invariant under the *duality* interchanges $a, b, c \leftrightarrow A, B, C$; $\phi_a, \phi_b, \phi_c \leftrightarrow \Phi_A, \Phi_B, \Phi_C$ and $P \leftrightarrow J, T \leftrightarrow I$; this duality follows from the automorphism $\mathcal{D}$ (3.6) in the family of CKD algebras $\mathcal{D} : P_1 \leftrightarrow -J, \mathcal{D} : T_1 \leftrightarrow -I$. The equations resemble their real analogues: real rotations e^{AJ_A} or translations e^{aP_a} are replaced by the 'complete' products $e^{AJ_A}\, e^{\Phi_A I_A}$ or $e^{aP_a}\, e^{\phi_a T_a}$, and each equation is a pair relating both generators of 'complete' transformations. As in [1] we will refer to them as $P_b(P_a), T_b(T_a)$, or $P_a(P_b), T_a(T_b)$ etc. By cyclic substitution in the three pairs of equations $P_a(P_c), T_a(T_c)$; $P_c(P_b), T_c(T_b)$ and $P_b(P_a), T_b(T_a)$ and its dual proccess, we find the identities

$$e^{BJ_B}\, e^{\Phi_B I_B}\, e^{-AJ_A}\, e^{-\Phi_A I_A}\, e^{CJ_C}\, e^{\Phi_C I_C} \begin{pmatrix}P_a\\T_a\end{pmatrix} e^{-\Phi_C I_C}\, e^{-CJ_C}\, e^{\Phi_A I_A}\, e^{AJ_A}\, e^{-\Phi_B I_B}\, e^{-BJ_B} = \begin{pmatrix}P_a\\T_a\end{pmatrix} \tag{5.4}$$

$$e^{-aP_a}\, e^{-\phi_a T_a}\, e^{cP_c}\, e^{\phi_c T_c}\, e^{bP_b}\, e^{\phi_b T_b} \begin{pmatrix}J_C\\I_C\end{pmatrix} e^{-\phi_b T_b}\, e^{-bP_b}\, e^{-\phi_c T_c}\, e^{-cP_c}\, e^{\phi_a T_a}\, e^{aP_a} = \begin{pmatrix}J_C\\I_C\end{pmatrix} \tag{5.5}$$

which can be written alternatively as

$$\begin{aligned} &e^{BJ_B}\, e^{\Phi_B I_B}\, e^{-AJ_A}\, e^{-\Phi_A I_A}\, e^{CJ_C}\, e^{\Phi_C I_C} \text{ must commute with } P_a \text{ and } T_a\\ &e^{-aP_a}\, e^{-\phi_a T_a}\, e^{cP_c}\, e^{\phi_c T_c}\, e^{bP_b}\, e^{\phi_b T_b} \text{ must commute with } J_C \text{ and } I_C. \end{aligned} \tag{5.6}$$

5.1. Loop equations

This self-dual approach affords some simple and apparently new results for certain 'loop' operators. The guideline is the pattern established in the real case, replacing translation or rotation generators T/J by its 'complete' versions $P, T/J, I$. We start with the equation which gives $P_c(P_b), T_c(T_b)$ in set (5.3), replace $\binom{P_c}{T_c}$ by $e^{-cP_c}\, e^{-\phi_c T_c} \binom{P_c}{T_c} e^{\phi_c T_c}\, e^{cP_c}$ and then substitute $P_c(P_a), T_c(T_a)$ from the compatibility equations to obtain

$$e^{-AJ_A}\, e^{-\Phi_A I_A} \begin{pmatrix}P_b\\T_b\end{pmatrix} e^{\Phi_A I_A}\, e^{AJ_A} = e^{-cP_c}\, e^{-\phi_c T_c}\, e^{-BJ_B}\, e^{-\Phi_B I_B} \begin{pmatrix}P_a\\T_a\end{pmatrix} e^{\Phi_B I_B}\, e^{BJ_B}\, e^{\phi_c T_c}\, e^{cP_c}. \tag{5.7}$$

We introduce $J_B(J_C), I_B(I_C)$, simplify and rearrange. Now we use $J_A(J_C), I_A(I_C)$, simplify again and finally substitute $P_b(P_a), T_b(T_a)$. This gives

$$\begin{aligned} &e^{BJ_C}\, e^{\Phi_B I_C}\, e^{-aP_a}\, e^{-\phi_a T_a}\, e^{cP_c}\, e^{\phi_c T_c}\, e^{bP_b}\, e^{\phi_b T_b}\, e^{-AJ_C}\, e^{-\Phi_A I_C}\, e^{CJ_C}\, e^{\Phi_C I_C} \begin{pmatrix}P_a\\T_a\end{pmatrix} e^{-\Phi_C I_C}\, e^{-CJ_C}\, e^{\Phi_A I_C}\\ &\quad\times e^{AJ_C}\, e^{-\phi_b T_b}\, e^{-bP_b}\, e^{-\phi_c T_c}\, e^{-cP_c}\, e^{\phi_a T_a}\, e^{aP_a}\, e^{-\Phi_B I_C}\, e^{-BJ_C} = \begin{pmatrix}P_a\\T_a\end{pmatrix}. \end{aligned} \tag{5.8}$$

Three *complete* translations along the triangle appear in the former relation in a single piece $e^{-aP_a}\, e^{-\phi_a T_a}\, e^{cP_c}\, e^{\phi_c T_c}\, e^{bP_b}\, e^{\phi_b T_b}$, while the three *complete* rotations are all about the base point C. Now we can go a bit further: (5.6) implies that $e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b}$ must commute with J_C, I_C, so it will commute with the complete rotation $e^{-\Phi_X I_C}\,e^{-XJ_C}$ about C for *any* values of X, Φ_X. Then we can commute the whole complete translation piece $e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b}$ in (5.8) with the rotations about C and collect these altogether; as both components of the complete rotation do commute, we get

$$e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b}\,e^{(-A+B+C)J_C}\,e^{(-\Phi_A+\Phi_B+\Phi_C)I_C} \text{ must commute with } P_a \text{ and } T_a. \tag{5.9}$$

We have already derived (see (5.6)) that $e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b}\,e^{XJ_C}\,e^{\Phi_X I_C}$ must commute with J_C and I_C for *any* 'complete angle' (X, Φ_X) . Since it also commutes with P_a, T_a for the special values $X = -A + B + C$, $\Phi_X = -\Phi_A + \Phi_B + \Phi_C$, we can conclude that

$$e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b}\,e^{(-A+B+C)J_C}\,e^{(-\Phi_A+\Phi_B+\Phi_C)I_C} = 1 \tag{5.10}$$

because the identity is the only element of ${}_\eta SU_{\kappa_1,\kappa_2}(3)$ commuting with two such pairs of generators as P_a, T_a and J_C, I_C. This equation can also be written as

$$e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b} = e^{-(-A+B+C)J_C}\,e^{-(-\Phi_A+\Phi_B+\Phi_C)I_C}. \tag{5.11}$$

A similar procedure (or direct use of (5.3) in (5.11)) allows us to analogously derive

$$\begin{aligned} e^{bP_b}\,e^{\phi_b T_b}\,e^{-aP_a}\,e^{-\phi_a T_a}\,e^{cP_c}\,e^{\phi_c T_c} &= e^{-(-A+B+C)J_A}\,e^{-(-\Phi_A+\Phi_B+\Phi_C)I_A} \\ e^{cP_c}\,e^{\phi_c T_c}\,e^{bP_b}\,e^{\phi_b T_b}\,e^{-aP_a}\,e^{-\phi_a T_a} &= e^{-(-A+B+C)J_B}\,e^{-(-\Phi_A+\Phi_B+\Phi_C)I_B}. \end{aligned} \tag{5.12}$$

Equations (5.11) or (5.12), to be called the *'complex Hermitian' point loop equations*, express the product of the three *complete* translations along the oriented sides of the triangle loop as a complete rotation about the loop base point. These equations are the closest 'complex Hermitian' analogues of the Gauss–Bonnet triangle theorem (see [1]) but we have found no reference to such a simple result in the literature. The explicit duality of the starting equations (5.3) under the interchanges $a, b, c \leftrightarrow A, B, C$ and $\phi_a, \phi_b, \phi_c \leftrightarrow \Phi_A, \Phi_B, \Phi_C$ and $P \leftrightarrow J, T \leftrightarrow I$ immediately implies that the dual process leads to the dual partners of (5.11), (5.12):

$$\begin{aligned} e^{-AJ_A}\,e^{-\Phi_A I_A}\,e^{CJ_C}\,e^{\Phi_C I_C}\,e^{BJ_B}\,e^{\Phi_B I_B} &= e^{-(-a+b+c)P_c}\,e^{-(-\phi_a+\phi_b+\phi_c)T_c} \\ e^{BJ_B}\,e^{\Phi_B I_B}\,e^{-AJ_A}\,e^{-\Phi_A I_A}\,e^{CJ_C}\,e^{\Phi_C I_C} &= e^{-(-a+b+c)P_a}\,e^{-(-\phi_a+\phi_b+\phi_c)T_a} \\ e^{CJ_C}\,e^{\Phi_C I_C}\,e^{BJ_B}\,e^{\Phi_B I_B}\,e^{-AJ_A}\,e^{-\Phi_A I_A} &= e^{-(-a+b+c)P_b}\,e^{-(-\phi_a+\phi_b+\phi_c)T_b}. \end{aligned} \tag{5.13}$$

5.2. The basic trigonometric identity

In any CKD 'complex Hermitian' space, each equation in (5.11), (5.12) and (5.13) contains the relationships between triangle sides, angles and phases which appear in them explicitly as canonical parameters, but also implicitly inside the generators of complete transformations. This implicit, hidden dependence can be eliminated leading to a new equation which contains much more explicitly the trigonometry of the space. The idea is to express *all* the generators as suitable conjugates of *one* set of primitive *independent* generators, the two pairs P_a, T_a and

J_C, I_C. Using (5.3) we *define* the remaining generators P_b, T_b; J_A, I_A; P_c, T_c; J_B, I_B in terms of the previous ones and sides, angles and angular phases as

$$\begin{aligned}
\begin{pmatrix} P_b \\ T_b \end{pmatrix} &:= \mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C} \begin{pmatrix} P_a \\ T_a \end{pmatrix} \mathrm{e}^{-\Phi_C I_C}\,\mathrm{e}^{-CJ_C} \\
\begin{pmatrix} J_A \\ I_A \end{pmatrix} &:= \mathrm{e}^{bP_b}\,\mathrm{e}^{\phi_b T_b} \begin{pmatrix} J_C \\ I_C \end{pmatrix} \mathrm{e}^{-\phi_b T_b}\,\mathrm{e}^{-bP_b} \\
\begin{pmatrix} P_c \\ T_c \end{pmatrix} &:= \mathrm{e}^{-AJ_A}\,\mathrm{e}^{-\Phi_A I_A} \begin{pmatrix} P_b \\ T_b \end{pmatrix} \mathrm{e}^{\Phi_A I_A}\,\mathrm{e}^{AJ_A} \\
\begin{pmatrix} J_B \\ I_B \end{pmatrix} &:= \mathrm{e}^{cP_c}\,\mathrm{e}^{\phi_c T_c} \begin{pmatrix} J_A \\ I_A \end{pmatrix} \mathrm{e}^{-\phi_c T_c}\,\mathrm{e}^{-cP_c}
\end{aligned} \tag{5.14}$$

which after full expansion and simplification give (note the highly ordered pattern)

$$\begin{aligned}
\begin{pmatrix} P_b \\ T_b \end{pmatrix} &:= \mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C} \begin{pmatrix} P_a \\ T_a \end{pmatrix} \mathrm{e}^{-\Phi_C I_C}\,\mathrm{e}^{-CJ_C} \\
\begin{pmatrix} J_A \\ I_A \end{pmatrix} &:= \mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C}\,\mathrm{e}^{bP_a}\,\mathrm{e}^{\phi_b T_a} \begin{pmatrix} J_C \\ I_C \end{pmatrix} \mathrm{e}^{-\phi_b T_a}\,\mathrm{e}^{-bP_a}\,\mathrm{e}^{-\Phi_C I_C}\,\mathrm{e}^{-CJ_C} \\
\begin{pmatrix} P_c \\ T_c \end{pmatrix} &:= \mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C}\,\mathrm{e}^{bP_a}\,\mathrm{e}^{\phi_b T_a}\,\mathrm{e}^{-AJ_C}\,\mathrm{e}^{-\Phi_A I_C} \begin{pmatrix} P_a \\ T_a \end{pmatrix} \mathrm{e}^{\Phi_A I_C}\,\mathrm{e}^{AJ_C}\,\mathrm{e}^{-\phi_b T_a}\,\mathrm{e}^{-bP_a}\,\mathrm{e}^{-\Phi_C I_C}\,\mathrm{e}^{-CJ_C} \\
\begin{pmatrix} J_B \\ I_B \end{pmatrix} &:= \mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C}\,\mathrm{e}^{bP_a}\,\mathrm{e}^{\phi_b T_a}\,\mathrm{e}^{-AJ_C}\,\mathrm{e}^{-\Phi_A I_C}\,\mathrm{e}^{\phi_c T_a}\,\mathrm{e}^{cP_a} \begin{pmatrix} J_C \\ I_C \end{pmatrix} \\
&\quad \times \mathrm{e}^{-\phi_c T_a}\,\mathrm{e}^{-cP_a}\,\mathrm{e}^{\Phi_A I_C}\,\mathrm{e}^{AJ_C}\,\mathrm{e}^{\phi_b T_a}\,\mathrm{e}^{-bP_a}\,\mathrm{e}^{-\Phi_C I_C}\,\mathrm{e}^{-CJ_C}.
\end{aligned} \tag{5.15}$$

Direct substitution in (5.11) and cancellations fully mimicking the pattern found in the real case (due to the commutativity of pairs in the complete transformations) give

$$\mathrm{e}^{-aP_a}\,\mathrm{e}^{-\phi_a T_a}\,\mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C}\,\mathrm{e}^{bP_a}\,\mathrm{e}^{\phi_b T_a}\,\mathrm{e}^{-AJ_C}\,\mathrm{e}^{-\Phi_A I_C}\,\mathrm{e}^{cP_a}\,\mathrm{e}^{\phi_c T_a}\,\mathrm{e}^{BJ_C}\,\mathrm{e}^{\Phi_B I_C} = 1. \tag{5.16}$$

The same process starting from any equation in (5.12) or (5.13) leads again to the same identity. This justifies calling (5.16) the *basic trigonometric equation*. We sum up in

Theorem 1. *Sides a, b, c, lateral phases ϕ_a, ϕ_b, ϕ_c, angles A, B, C and angular phases Φ_A, Φ_B, Φ_C of any triangle loop in the complex CKD space $\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2}$ are linked by a single group identity called the basic 'complex Hermitian' trigonometric identity*

$$\mathrm{e}^{-aP}\,\mathrm{e}^{-\phi_a T}\,\mathrm{e}^{CJ}\,\mathrm{e}^{\Phi_C I}\,\mathrm{e}^{bP}\,\mathrm{e}^{\phi_b T}\,\mathrm{e}^{-AJ}\,\mathrm{e}^{-\Phi_A I}\,\mathrm{e}^{cP}\,\mathrm{e}^{\phi_c T}\,\mathrm{e}^{BJ}\,\mathrm{e}^{\Phi_B I} = 1 \tag{5.17}$$

where P, T are the generators of translations and phase translations along any fixed fiducial geodesic g, and J, I are the generators of rotations and phase rotations about any fixed fiducial 'complex' line-geodesic G containing the 'complex' line l_g and about O.

Proof. A group motion can be used to move the triangle to a canonical position described before (5.1) for the flag $O \subset g \subset l \subset l_g$. Then the theorem statement is simply (5.16). □

Theorem 2. *Let us consider a triangle loop in the 'complex Hermitian' CKD space $\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2}$, and let P_a, P_b, P_c; T_a, T_b, T_c be the generators of translations and phase translations along the three triangle geodesic sides, whose lengths and lateral phases are a, b, c and ϕ_a, ϕ_b, ϕ_c. Let J_A, J_B, J_C; I_A, I_B, I_C be the generators of rotations and phase rotations about the three geodesic line vertices of the triangle, whose angles and angular phases are A, B, C and Φ_A, Φ_B, Φ_C. These quantities are related by two sets of identities, (5.11), (5.12) and (5.13)* *called the ‘complex Hermitian’ point loop and the ‘complex Hermitian’ line loop triangle equations, each equation being equivalent to the identity in theorem 1.*

Several points are worth highlighting. First, each term in the basic identity is either a complete translation along a fixed geodesic g (through O) or a complete rotation about O (along a fixed line-geodesic G) with canonical parameters the triangle sides, angles, lateral and angular phases. In the point loop or line loop equations, the transformations involved are the translations along the sides or the rotations about the vertices. Second, these equations are similar to the real case, with consistent replacement of every translation or rotation by its ‘complete’ version. Third, the (three) point loop equations and the (three) line loop equations are mutually dual sets; the single basic equation is self-dual. And fourth, these equations hold in the same explicit form for *all* 27 2D ‘complex Hermitian’ CKD geometries, as neither CD label η nor a CK one κ_1, κ_2 ever appears in them.

6. The basic equations of trigonometry for any ‘complex Hermitian’ 2D Cayley–Klein–Dickson space

To obtain trigonometric equations for the ‘complex Hermitian’ CKD space, we start with the basic trigonometric identity (5.17) for the triangle in its canonical position, so P_a, T_a and J_C, I_C can be taken exactly as P_1, T_1 (denoted simply by P, T) and J, I, written as

$$\mathrm{e}^{-AJ}\,\mathrm{e}^{-\Phi_A I}\,\mathrm{e}^{cP}\,\mathrm{e}^{\phi_c T}\,\mathrm{e}^{BJ}\,\mathrm{e}^{\Phi_B I} = \mathrm{e}^{-bP}\,\mathrm{e}^{-\phi_b T}\,\mathrm{e}^{-CJ}\,\mathrm{e}^{-\Phi_C I}\,\mathrm{e}^{aP}\,\mathrm{e}^{\phi_a T} \tag{6.1}$$

and then write it in the fundamental 3D vector representation of the motion groups (3.7) and (3.9) by 3×3 complex matrices, giving rise to nine ‘complex’ identities:

$$\begin{aligned}
&C_{\kappa_1}(c)\,\mathrm{e}^{\mathrm{i}\frac{-2\Phi_A+2\Phi_B+\phi_c}{3}} = C_{\kappa_1}(a)C_{\kappa_1}(b)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b-2\Phi_C}{3}} + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b)C_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b+\Phi_C}{3}}\\
&C_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{-2\phi_a+2\phi_b+\Phi_C}{3}} = C_{\kappa_2}(A)C_{\kappa_2}(B)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B-2\phi_c}{3}} + \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(B)C_{\kappa_1}(c)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B+\phi_c}{3}}\\
&S_{\kappa_1}(c)S_{\kappa_2}(A)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A+2\Phi_B+\phi_c}{3}} = S_{\kappa_1}(a)S_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a+2\phi_b+\Phi_C}{3}}\\
&S_{\kappa_1}(c)S_{\kappa_2}(B)\,\mathrm{e}^{\mathrm{i}\frac{-2\Phi_A-\Phi_B+\phi_c}{3}} = S_{\kappa_1}(b)S_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{-2\phi_a-\phi_b+\Phi_C}{3}}\\
&S_{\kappa_1}(c)C_{\kappa_2}(A)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A+2\Phi_B+\phi_c}{3}} = -C_{\kappa_1}(a)S_{\kappa_1}(b)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b-2\Phi_C}{3}} + S_{\kappa_1}(a)C_{\kappa_1}(b)C_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b+\Phi_C}{3}}\\
&S_{\kappa_1}(c)C_{\kappa_2}(B)\,\mathrm{e}^{\mathrm{i}\frac{-2\Phi_A-\Phi_B+\phi_c}{3}} = C_{\kappa_1}(b)S_{\kappa_1}(a)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b-2\Phi_C}{3}} - S_{\kappa_1}(b)C_{\kappa_1}(a)C_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b+\Phi_C}{3}}\\
&S_{\kappa_2}(C)C_{\kappa_1}(a)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a+2\phi_b+\Phi_C}{3}} = -C_{\kappa_2}(A)S_{\kappa_2}(B)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B-2\phi_c}{3}} + S_{\kappa_2}(A)C_{\kappa_2}(B)C_{\kappa_1}(c)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B+\phi_c}{3}}\\
&S_{\kappa_2}(C)C_{\kappa_1}(b)\,\mathrm{e}^{\mathrm{i}\frac{-2\phi_a-\phi_b+\Phi_C}{3}} = C_{\kappa_2}(B)S_{\kappa_2}(A)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B-2\phi_c}{3}} - S_{\kappa_2}(B)C_{\kappa_2}(A)C_{\kappa_1}(c)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B+\phi_c}{3}}\\
&\kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(B)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B-2\phi_c}{3}} + C_{\kappa_2}(A)C_{\kappa_2}(B)C_{\kappa_1}(c)\,\mathrm{e}^{\mathrm{i}\frac{\Phi_A-\Phi_B+\phi_c}{3}}\\
&\qquad = \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b-2\Phi_C}{3}} + C_{\kappa_1}(a)C_{\kappa_1}(b)C_{\kappa_2}(C)\,\mathrm{e}^{\mathrm{i}\frac{\phi_a-\phi_b+\Phi_C}{3}}
\end{aligned} \tag{6.2}$$

which contain the trigonometry of the space $\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2} = {}_\eta SU_{\kappa_1,\kappa_2}(3)/({}_\eta U(1) \otimes {}_\eta SU_{\kappa_2}(2))$.

Each equation in this set either is self-dual or appears in a mutually dual pair; this follows from the self-duality of the starting equation (recall along this section i denotes the imaginary unit of the CD ‘complex’ numbers $\mathbb{C}_\eta$ with $\mathrm{i}^2 = -\eta$, see (4.4)). The association between sides a, b, c (respectively angles A, B, C) and the labels κ_1 (respectively κ_2) found in the equations of the real space $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$ extends to the Hermitian complex analogues. The lateral ϕ_a, ϕ_b, ϕ_c and angular phases Φ_A, Φ_B, Φ_C have η as its label. The elements $(-a, -\phi_a, -A, -\Phi_A)$ always appear in the equations with a minus sign as compared with (b, ϕ_b, B, Φ_B), (c, ϕ_c, C, Φ_C); this follows from the structure of the basic equation (6.1), where the side a and vertex A are traversed or rotated backwards.

The set (6.2) is equivalent to two other similar sets, obtained by starting with the basic identity split into the two equivalent forms $Cb\overline{A} = cB\overline{a}$ and $b\overline{A}c = B\overline{a}C$ instead of $\overline{A}cB = \bar{b}\overline{C}a$ as in (6.1). To present all these equations in a concise way, a compact notation following the pattern explained in [1] is useful: sides and angles, lateral phases and angular phases will be denoted as $x_i, X_I, \phi_i, \Phi_I, i, I = 1, 2, 3$:

$$\begin{array}{llllll} x_1 = -a & x_2 = b & x_3 = c & X_1 = -A & X_2 = B & X_3 = C \\ \phi_1 = -\phi_a & \phi_2 = \phi_b & \phi_3 = \phi_c & \Phi_1 = -\Phi_A & \Phi_2 = \Phi_B & \Phi_3 = \Phi_C. \end{array} \tag{6.3}$$

The built-in minus sign for $i = I = 1$ is natural when the triangle is considered as a point loop with the side a, ϕ_a traversed backwards, or as a side loop with the angle A, Φ_A rotated backwards; further it renders a uniform appearance to equations (6.2). The basic equation (5.17), for *any* cyclic permutation $i = I, j = J, k = K$ of 123 is

$$\mathrm{e}^{x_i P}\,\mathrm{e}^{\phi_i T}\,\mathrm{e}^{X_K J}\,\mathrm{e}^{\Phi_K I}\,\mathrm{e}^{x_j P}\,\mathrm{e}^{\phi_j T}\mathrm{e}^{X_I J}\,\mathrm{e}^{\Phi_I I}\,\mathrm{e}^{x_k P}\,\mathrm{e}^{\phi_k T}\,\mathrm{e}^{X_J J}\,\mathrm{e}^{\Phi_J I} = 1. \tag{6.4}$$

From now on, we will adopt this convention which makes equations of trigonometry invariant under cyclic permutations of the 'oriented' complete sides x_i, ϕ_i and angles X_I, Φ_I.

6.1. The trigonometric equations in the 'Cartan sector'

Each equation in (6.2) is 'complex', and phases appear through unimodular 'complex' factors $\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\Phi}$, while 'pure' sides and angles x_i, X_I appear through their real labelled sines or cosines. Equations in the third line split into modulus and argument; this last is

$$-\phi_j + 2\phi_k + \Phi_I = -\Phi_J + 2\Phi_K + \phi_i. \tag{6.5}$$

Writing the same equation for the choice of indices $i, j, k \to j, k, i$ and comparing, we get

$$\phi_i - \Phi_I = \phi_j - \Phi_J. \tag{6.6}$$

These three equations, only two of which are independent, are self-dual and hold for all the 'complex' CKD spaces. Two consequences of these very simple *linear* relations are

$$-\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c \tag{6.7}$$

$$-\Phi_A + \phi_b + \phi_c = -\phi_a + \Phi_B + \phi_c = -\phi_a + \phi_b + \Phi_C \tag{6.8}$$

and the common values in these formulae, we will call Ω and ω, turn out to be the quantities which for $\mathbb{C}P^2$ were introduced by Blaschke–Terheggen:

$$\Omega := \Phi_I + \Phi_J + \phi_k \qquad \omega := \phi_i + \phi_j + \Phi_K. \tag{6.9}$$

Thus there is a sector of Hermitian trigonometry involving *only* phases and completely decoupled from sides x_i and angles X_I. This sector holds in *exactly the same form* in the twenty seven 'complex' CKD spaces, as no explicit labels $\eta; \kappa_1, \kappa_2$ appear. Since the triangle invariants ϕ_i, Φ_I are related to the Cartan subalgebra, we will call these equations the 'Cartan' sector of 'complex Hermitian' trigonometry. This 'Cartan sector' has no analogue in the trigonometry of real spaces, and their equations are purely linear witnessing the Abelian character of Cartan subalgebra. Of course, all equations in this section must be understood as $\mathrm{mod}\left(\frac{2\pi}{\sqrt{\eta}}\right)$ when $\eta > 0$, as 'phases' are then ordinary circular phases.

6.2. The complete set of ‘complex Hermitian’ trigonometric equations

Now by exploiting the ‘Cartan’ relations between phases (6.6), and introducing explicitly the invariants Ω, ω, it becomes possible to simplify equations (6.2) by multiplying each one of them by some suitably chosen unimodular ‘complex’ factor. This leads to the *full* set of trigonometric equations coming from the basic trigonometric group identity as

$$
\begin{array}{ll}
0ij \equiv 0IJ & \Phi_I - \phi_i = \Phi_J - \phi_j \quad (\Rightarrow \Omega := \Phi_I + \Phi_J + \phi_k,\ \omega := \phi_i + \phi_j + \Phi_K) \\
1i & C_{\kappa_1}(x_i)\, e^{i\Omega} = C_{\kappa_1}(x_j) C_{\kappa_1}(x_k) - \kappa_1 S_{\kappa_1}(x_j) S_{\kappa_1}(x_k) C_{\kappa_2}(X_I)\, e^{i\Phi_I} \\
1I & C_{\kappa_2}(X_I)\, e^{i\omega} = C_{\kappa_2}(X_J) C_{\kappa_2}(X_K) - \kappa_2 S_{\kappa_2}(X_J) S_{\kappa_2}(X_K) C_{\kappa_1}(x_i)\, e^{i\phi_i} \\
2ij \equiv 2IJ & \dfrac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} = \dfrac{S_{\kappa_1}(x_j)}{S_{\kappa_2}(X_J)} \\
3iJ & S_{\kappa_1}(x_i) C_{\kappa_2}(X_J)\, e^{i\phi_k} = -C_{\kappa_1}(x_j) S_{\kappa_1}(x_k)\, e^{-i\Phi_I} - S_{\kappa_1}(x_j) C_{\kappa_1}(x_k) C_{\kappa_2}(X_I) \\
3Ij & S_{\kappa_2}(X_I) C_{\kappa_1}(x_j)\, e^{i\Phi_K} = -C_{\kappa_2}(X_J) S_{\kappa_2}(X_K)\, e^{-i\phi_i} - S_{\kappa_2}(X_J) C_{\kappa_2}(X_K) C_{\kappa_1}(x_i) \\
4ij \equiv 4IJ & -\kappa_1 S_{\kappa_1}(x_i) S_{\kappa_1}(x_j) + C_{\kappa_1}(x_i) C_{\kappa_1}(x_j) C_{\kappa_2}(X_K)\, e^{i\Phi_K} \\
& \qquad = -\kappa_2 S_{\kappa_2}(X_I) S_{\kappa_2}(X_J) + C_{\kappa_2}(X_I) C_{\kappa_2}(X_J) C_{\kappa_1}(x_k)\, e^{i\phi_k}.
\end{array}
\tag{6.10}
$$

These equations will be referred to by a tag, and either are self-dual (for instance $2ij \equiv 2IJ$) or appear in mutually dual pairs (as $1i, 1I$). Equations with tag 0 are in the ‘Cartan’ sector. The remaining tags are intentionally made to match those used in [1] and the equations with the same tags are in most respects the closest ‘complex Hermitian’ analogues to the equations found in the real case, as far as mutual relations, dependence or independence etc are concerned. Trigonometry of real spaces thus provides a first guide in the exploration of the whole forest of ‘complex Hermitian’ trigonometric equations.

6.3. The ‘complex Hermitian’ trigonometric bestiarium

Taking (6.10) as a starting point, we now perform a fully explicit study of ‘complex Hermitian’ trigonometry. As the scheme enjoys self-duality, those equations which are not self-dual will have a dual partner, obtained by exchange in capitalization of names and indices: $x \leftrightarrow X, \phi \leftrightarrow \Phi, i \leftrightarrow I$ and $\kappa_1 \leftrightarrow \kappa_2$; in these cases we will only sketch the derivation of one member of the dual pair, but we will write each pair together, to emphasize self-duality as the main trait of this approach. The label η does not change under duality.

These equations will hold for *all 27* ‘complex’ CKD spaces with arbitrary $\eta; \kappa_1, \kappa_2$. In the degenerate cases $\kappa_1 = 0$ (flat ‘complex Hermitian’ spaces) and/or $\eta = 0$ or $\kappa_2 = 0$ (degenerate ‘complex’ ‘Hermitian’ metric) some equations may collapse or even reduce to trivial identities; these cases will be discussed later but for the moment we will stay in the general situation where $\eta; \kappa_1, \kappa_2$ are assumed to have *any* values. All equations found in the literature for the elliptic (hyperbolic) complex Hermitian spaces will follow from this set after we specialize $\eta = 1; \kappa_1 = 1$ $(\kappa_1 = -1)$, $\kappa_2 = 1$.

- *Cartan sector* equations $0IJ \equiv 0ij$ will be called the *‘complex Hermitian’ phase theorem*. They are self-dual, involve only the triangle Cartan invariants and allow the introduction of two symmetric triangle invariants Ω and ω after (6.9). There are two independent equations, thus *four* independent quantities among the six lateral/angular phases:

$$
0IJ \equiv 0ij \qquad \phi_i - \Phi_I = \phi_j - \Phi_J = \phi_k - \Phi_K = \omega - \Omega. \tag{6.11}
$$

- The equations $2iJ \equiv 2jI$, taken together will be called the *Hermitian sine theorem*:

$$2IJ \equiv 2ij \qquad \frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} = \frac{S_{\kappa_1}(x_j)}{S_{\kappa_2}(X_J)} = \frac{S_{\kappa_1}(x_k)}{S_{\kappa_2}(X_K)}. \tag{6.12}$$

The Hermitian phase theorem (6.11) and sine theorem (6.12) are the modulus and argument of the same 'complex' equality, the *'complex' Hermitian sine theorem*.

- Each of the *'complex Hermitian' cosine theorems* $1i$ and $1I$ is a 'complex' equation. By splitting the Hermitian cosine theorem $1i$ into real and imaginary parts, we get

$$1i \qquad \begin{aligned} C_{\kappa_1}(x_i)C_\eta(\Omega) &= C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) - \kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)C_\eta(\Phi_I) \\ C_{\kappa_1}(x_i)S_\eta(\Omega) &= -\kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)S_\eta(\Phi_I) \end{aligned} \tag{6.13}$$

called real and imaginary Hermitian cosine laws (for sides). Their duals are the real and imaginary Hermitian dual cosine laws (for angles):

$$1I \qquad \begin{aligned} C_{\kappa_2}(X_I)C_\eta(\omega) &= C_{\kappa_2}(X_J)C_{\kappa_2}(X_K) - \kappa_2 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)C_\eta(\phi_i) \\ C_{\kappa_2}(X_I)S_\eta(\omega) &= -\kappa_2 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)S_\eta(\phi_i). \end{aligned} \tag{6.14}$$

- By equating the modulus of both sides of the Hermitian cosine theorem $1i$ we get:

$$\begin{aligned} C^2_{\kappa_1}(x_i) = {} & \big(C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) - \kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)C_\eta(\Phi_I)\big)^2 \\ & + \eta\kappa_1^2 S^2_{\kappa_1}(x_j)S^2_{\kappa_1}(x_k)C^2_{\kappa_2}(X_I)S^2_\eta(\Phi_I) \end{aligned} \tag{6.15}$$

which for $\mathbb{C}P^2$ is the Shirokov–Rosenfeld [9] cosine theorem, yet expressed instead in terms of our angular variables X_I and Φ_I. This theorem admits another form, starting from $C_{\kappa_1}(2x_i) + 1 = 2C^2_{\kappa_1}(x_i)$, substituting (6.15) and expanding the squared sines of sides,

$$\begin{aligned} C_{\kappa_1}(2x_i) = {} & C_{\kappa_1}(2x_j)C_{\kappa_1}(2x_k) - \kappa_1 S_{\kappa_1}(2x_j)S_{\kappa_1}(2x_k)C_{\kappa_2}(X_I)C_\eta(\Phi_I) \\ & - 2\kappa_1^2\kappa_2 S^2_{\kappa_1}(x_j)S^2_{\kappa_1}(x_k)S^2_{\kappa_2}(X_I) \end{aligned} \tag{6.16}$$

reducing to (2.13) for $\mathbb{C}P^2$ (recall our choice of external angles at A; this is not done in (2.13)). The duals are

$$\begin{aligned} C^2_{\kappa_2}(X_I) = {} & \big(C_{\kappa_2}(X_J)C_{\kappa_2}(X_K) - \kappa_2 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)C_\eta(\phi_i)\big)^2 \\ & + \eta\kappa_2^2 S^2_{\kappa_2}(X_J)S^2_{\kappa_2}(X_K)C^2_{\kappa_1}(x_i)S^2_\eta(\phi_i) \end{aligned} \tag{6.17}$$

$$\begin{aligned} C_{\kappa_2}(2X_I) = {} & C_{\kappa_2}(2X_J)C_{\kappa_2}(2X_K) - \kappa_2 S_{\kappa_2}(2X_J)S_{\kappa_2}(2X_K)C_{\kappa_1}(x_i)C_\eta(\phi_j) \\ & - 2\kappa_1\kappa_2^2 S^2_{\kappa_2}(X_J)S^2_{\kappa_2}(X_K)S^2_{\kappa_1}(x_i). \end{aligned} \tag{6.18}$$

- By building up the term $\kappa_1 S_{\kappa_1}(2x_j)S_{\kappa_1}(2x_k)C_{\kappa_2}(X_I)C_\eta(\Phi_I)$ in (6.13) and substituting into (6.16), expanding and simplifying we obtain

$$C^2_{\kappa_1}(x_i) = -C^2_{\kappa_1}(x_j)C^2_{\kappa_1}(x_k) + \kappa_1^2 S^2_{\kappa_1}(x_j)S^2_{\kappa_1}(x_k)C^2_{\kappa_2}(X_I) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_\eta(\Omega) \tag{6.19}$$

reducing for $\mathbb{C}P^2$ to the Blaschke–Terheggen cosine theorem for sides (2.8). Its dual, reducing for $\mathbb{C}P^2$ to (2.9) is

$$\begin{aligned} C^2_{\kappa_1}(X_I) = {} & -C^2_{\kappa_2}(X_J)C^2_{\kappa_2}(X_K) + \kappa_2^2 S^2_{\kappa_2}(X_J)S^2_{\kappa_2}(X_K)C^2_{\kappa_1}(x_i) \\ & + 2C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)C_\eta(\omega). \end{aligned} \tag{6.20}$$

- Multiplying both sides of (6.12) by $1/S_\eta(\Omega)$ and using the second equation in (6.13) we obtain

$$\frac{S_{\kappa_1}(2x_i)}{S_\eta(\Phi_I)C_{\kappa_2}(X_I)} = \frac{S_{\kappa_1}(2x_j)}{S_\eta(\Phi_J)C_{\kappa_2}(X_J)} \tag{6.21}$$

which in $\mathbb{C}P^2$ reduces to the Shirokov–Rosenfeld double sine theorem (2.11) after changing to the angular variables used by SR. Its dual is

$$\frac{S_{\kappa_2}(2X_I)}{S_\eta(\phi_i)C_{\kappa_1}(x_i)} = \frac{S_{\kappa_2}(2X_J)}{S_\eta(\phi_j)C_{\kappa_1}(x_j)}. \tag{6.22}$$

- By multiplying (6.21) and (6.22) we get the self-dual equation

$$\frac{S_{\kappa_1}(x_i)S_{\kappa_1}(X_I)}{S_\eta(\phi_i)S_\eta(\Phi_I)} = \frac{S_{\kappa_1}(x_j)S_{\kappa_1}(X_J)}{S_\eta(\phi_j)S_\eta(\Phi_J)}. \tag{6.23}$$

- By taking the quotient between the double sine theorem (6.21) and the sine theorem (6.12) we get

$$\frac{C_{\kappa_1}(x_i)T_{\kappa_2}(X_I)}{S_\eta(\Phi_I)} = \frac{C_{\kappa_1}(x_j)T_{\kappa_2}(X_J)}{S_\eta(\Phi_J)} \tag{6.24}$$

whose dual is

$$\frac{C_{\kappa_2}(X_I)T_{\kappa_1}(x_i)}{S_\eta(\phi_i)} = \frac{C_{\kappa_2}(X_J)T_{\kappa_1}(x_j)}{S_\eta(\phi_j)}. \tag{6.25}$$

- Another set of equations derive from the equations with tags $3iJ$ and $3Ij$. In particular, by splitting them into their real and imaginary parts we obtain

$$3iJ \qquad \begin{aligned} &S_{\kappa_1}(x_i)C_{\kappa_2}(X_J)C_\eta(\phi_k) = -C_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_\eta(\Phi_I) - S_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_{\kappa_2}(X_I)\\ &S_{\kappa_1}(x_i)C_{\kappa_2}(X_J)S_\eta(\phi_k) = C_{\kappa_1}(x_j)S_{\kappa_1}(x_k)S_\eta(\Phi_I)\end{aligned} \tag{6.26}$$

$$3Ij \qquad \begin{aligned} &S_{\kappa_2}(X_I)C_{\kappa_1}(x_j)C_\eta(\Phi_K) = -C_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_\eta(\phi_i) - S_{\kappa_2}(X_J)C_{\kappa_2}(X_K)C_{\kappa_1}(x_i)\\ &S_{\kappa_2}(X_I)C_{\kappa_1}(x_j)S_\eta(\Phi_K) = C_{\kappa_2}(X_J)S_{\kappa_2}(X_K)S_\eta(\phi_i).\end{aligned} \tag{6.27}$$

- The same splitting for the equations $4ij \equiv 4IJ$ leads to the pair of self-dual equations:

$$4ij \equiv 4IJ \qquad \begin{aligned} &-\kappa_2 S_{\kappa_2}(X_I)S_{\kappa_2}(X_J) + C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_1}(x_k)C_\eta(\phi_k)\\ &\qquad = -\kappa_1 S_{\kappa_1}(x_i)S_{\kappa_1}(x_j) + C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_2}(X_K)C_\eta(\Phi_K)\\ &C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_1}(x_k)S_\eta(\phi_k) = C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_2}(X_K)S_\eta(\Phi_K).\end{aligned} \tag{6.28}$$

- Starting from the real and imaginary parts of the 'complex Hermitian' cosine theorem (6.13), expanding the trigonometric functions of $\Omega = \Phi_I + \phi_j + \Phi_K$ by considering it as a sum of two phases and eliminating the term containing $C_\eta(\Phi_I + \phi_j)$ we get

$$-\frac{C_{\kappa_1}(x_i)}{S_\eta(\Phi_I)} = \frac{C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)}{S_\eta(\phi_j + \Phi_K)} = \frac{C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)}{S_\eta(\Omega - \Phi_I)}. \tag{6.29}$$

Its dual is

$$-\frac{C_{\kappa_2}(X_I)}{S_\eta(\phi_i)} = \frac{C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)}{S_\eta(\Phi_J + \phi_k)} = \frac{C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)}{S_\eta(\omega - \phi_i)} \tag{6.30}$$

where we have used the relations $\Phi_J + \phi_k = \Omega - \Phi_I = \omega - \phi_i$ which follow from the equations in the 'Cartan' sector and the definitions of Ω and ω.

- By dividing equation (6.24) by (6.29) we get

$$-\frac{T_{\kappa_2}(X_I)}{S_\eta(\Phi_I)} = \frac{T_{\kappa_2}(X_K)C_{\kappa_1}(x_j)}{S_\eta(\Phi_I+\phi_j)} = \frac{T_{\kappa_2}(X_K)C_{\kappa_1}(x_j)}{S_\eta(\Omega-\Phi_K)} \tag{6.31}$$

whose form for another suitable choice of indices is

$$-\frac{T_{\kappa_2}(X_K)}{S_\eta(\Phi_K)} = \frac{T_{\kappa_2}(X_J)C_{\kappa_1}(x_j)}{S_\eta(\Phi_K+\phi_j)} = \frac{T_{\kappa_2}(X_J)C_{\kappa_1}(x_j)}{S_\eta(\Omega-\Phi_I)}. \tag{6.32}$$

The duals of these equations are

$$-\frac{T_{\kappa_1}(x_i)}{S_\eta(\phi_i)} = \frac{T_{\kappa_1}(x_k)C_{\kappa_2}(X_J)}{S_\eta(\phi_i+\Phi_J)} = \frac{T_{\kappa_1}(x_k)C_{\kappa_2}(X_J)}{S_\eta(\omega-\phi_k)} \tag{6.33}$$

$$-\frac{T_{\kappa_1}(x_k)}{S_\eta(\phi_k)} = \frac{T_{\kappa_1}(x_j)C_{\kappa_2}(X_J)}{S_\eta(\phi_k+\Phi_J)} = \frac{T_{\kappa_1}(x_j)C_{\kappa_2}(X_J)}{S_\eta(\omega-\phi_i)}. \tag{6.34}$$

- By eliminating the angles X_I, X_K between (6.31) and (6.32),

$$C^2_{\kappa_1}(x_k) = \frac{S_\eta(\Phi_I+\phi_k)S_\eta(\Phi_J+\phi_k)}{S_\eta(\Phi_I)S_\eta(\Phi_J)} = \frac{S_\eta(\Omega-\Phi_I)S_\eta(\Omega-\Phi_J)}{S_\eta(\Phi_I)S_\eta(\Phi_J)} \tag{6.35}$$

whose dual is

$$C^2_{\kappa_2}(X_K) = \frac{S_\eta(\phi_i+\Phi_K)S_\eta(\phi_j+\Phi_K)}{S_\eta(\phi_i)S_\eta(\phi_j)} = \frac{S_\eta(\omega-\phi_i)S_\eta(\omega-\phi_j)}{S_\eta(\phi_i)S_\eta(\phi_j)}. \tag{6.36}$$

These equations give the cosine of each side (angle) in terms of the angular (lateral) phases *only*. They somehow resemble real trigonometry Euler equations for the cosine of half the sides (angles) in terms of angles (sides), but here pure sides (angles) are however given in terms of angular phases and Ω (lateral phases and ω).

- Using expansion of sines of sums or differences and elementary manipulation, we finally get the expression for the squared sines of the sides, which in spite of the presence of κ_1, κ_2 in denominators are still meaningful when $\kappa_1 \to 0$ or $\kappa_2 \to 0$ as we will see shortly:

$$S^2_{\kappa_1}(x_k) = -\frac{S_\eta(\phi_k)\frac{S_\eta(\Omega)}{\kappa_1}}{S_\eta(\Phi_I)S_\eta(\Phi_J)} \qquad S^2_{\kappa_2}(X_K) = -\frac{S_\eta(\Phi_K)\frac{S_\eta(\omega)}{\kappa_2}}{S_\eta(\phi_i)S_\eta(\phi_j)}. \tag{6.37}$$

6.4. Loop excesses and symplectic area and co-area

For real CK spaces, the angular excess $\Delta := -A + B + C$ shares three properties: Δ goes to zero with κ_1, it is *proportional* (coefficient κ_1) to triangle area, and satisfies triangular Gauss–Bonnet type equations. Dually, the lateral excess $\delta := -\Phi_A + \Phi_B + \Phi_C$ is proportional to the co-area, vanishes with κ_2, and satisfies dual Gauss–Bonnet type equations.

In the 'Hermitian' case, we may define

$$\Delta := X_I + X_J + X_K = -A + B + C \qquad \Delta_\Phi := \Phi_I + \Phi_J + \Phi_K = -\Phi_A + \Phi_B + \Phi_C \tag{6.38}$$

$$\delta := x_i + x_j + x_k = -a + b + c \qquad \delta_\phi := \phi_i + \phi_j + \phi_k = -\phi_a + \phi_b + \phi_c \tag{6.39}$$

which will be called, respectively, the *(Hermitian) angular excess*, *angular phase excess* and *lateral excess*, *lateral phase excess* of the triangle loop.

From the 'Hermitian' analogue of the Gauss–Bonnet equations (5.11), (5.12) the *complete* angular excess (Δ, Δ_Φ) fits into the view of the (oriented) total complete angle turned by the geodesic line loop when we successively perform the complete translations along the three sides. And dually, the *'complex Hermitian' line loop equations* (5.13) give the product of the three complete rotations about the three vertices as a complete translation along the base line of the loop, with (δ, δ_ϕ) as parameters. In this sense, both pairs (Δ, Δ_Φ), (δ, δ_ϕ) inherit the property of the real excesses linked to the Gauss–Bonnet equations.

A natural question is whether these excesses also vanish with κ_1 or κ_2. The answer is they do not, but there are some similar combinations which do; these comprise another type of excess. We first recall their definitions,

$$\Omega := \Phi_I + \Phi_J + \phi_k \qquad \omega := \phi_i + \phi_j + \Phi_K \tag{6.40}$$

displaying the nature of these invariants Ω (ω) as a kind of *mixed phase excess*, with dominance of angular (respectively lateral) phases. For $\mathbb{C}P^2$ these coincide with the Blaschke–Terheggen invariants Ω or ω, introduced in a completely different way. The departure from the BT notation ω, τ to our Ω, ω conforms to the typographical convention upper/lower case in order to stress duality and to convey the dominance of either upper case or lower case phases. These are related to the two angular and lateral phase excesses Δ_Φ, δ_ϕ by

$$\Delta_\Phi = 2\Omega - \omega \qquad \delta_\phi = 2\omega - \Omega. \tag{6.41}$$

We now see how these mixed excesses actually go to zero with κ_1, κ_2; this follows directly from the equations already derived. In the real case, the three cosine equations $1i$ $(1I)$ turn into trivial identities when $\kappa_1 = 0\,(\kappa_2 = 0)$. In the 'complex Hermitian' case, the *three 'complex'* equations $1i$ $(1I)$, which are independent when $\kappa_1 \neq 0\,(\kappa_2 \neq 0)$, collapse when $\kappa_1 = 0\,(\kappa_2 = 0)$ into a *single real* equation in the 'Cartan' sector, and as far as pure sides and angles are concerned become trivial:

$$1i \quad \text{when} \quad \kappa_1 = 0 \qquad e^{i\Omega} = C_\eta(\Omega) + iS_\eta(\Omega) = 1 \quad \text{implying} \quad C_\eta(\Omega) = 1 \quad S_\eta(\Omega) = 0 \tag{6.42}$$

$$1I \quad \text{when} \quad \kappa_2 = 0 \qquad e^{i\omega} = C_\eta(\omega) + iS_\eta(\omega) = 1 \quad \text{implying} \quad C_\eta(\omega) = 1 \quad S_\eta(\omega) = 0. \tag{6.43}$$

The behaviour of the quotient $\frac{S_\eta(\Omega)}{\kappa_1}$ as $\kappa_1 \to 0$ can be derived both from the imaginary part of the Hermitian cosine theorem (6.13) and from equations (6.37),

$$\frac{S_\eta(\Omega)}{\kappa_1} = -\frac{S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)S_\eta(\Phi_I)}{C_{\kappa_1}(x_i)} = -\frac{S_\eta(\Phi_I)S_\eta(\Phi_J)S^2_{\kappa_1}(x_k)}{S_\eta(\phi_k)} \tag{6.44}$$

and since this quotient remains *finite* as $\kappa_1 \to 0$, Ω behaves like the angular excess Δ in the CK real spaces. Dually, ω behaves as the real pure lateral excess δ:

$$\frac{S_\eta(\omega)}{\kappa_2} = -\frac{S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)S_\eta(\phi_i)}{C_{\kappa_2}(X_I)} = -\frac{S_\eta(\phi_i)S_\eta(\phi_j)S^2_{\kappa_2}(X_K)}{S_\eta(\Phi_K)}. \tag{6.45}$$

The real excesses Δ, δ are *proportional*, with coefficients κ_1 and κ_2, to the triangle area and co-area respectively. In the elliptic Hermitian space $\mathbb{C}P^2$ Hangan and Masala [24] found for the *symplectic* triangle area $\mathcal{S}$ the relation $\mathcal{S} = -\Omega/2$ (the minus sign comes form their opposite definition of the symplectic form). For any member of the CKD family of the 'complex Hermitian' spaces, the *definitions* for triangle symplectic area and co-area

$$\mathcal{S} := \frac{\Omega}{2\kappa_1} \qquad s := \frac{\omega}{2\kappa_2} \tag{6.46}$$

(note the factor 2) are in full agreement with the standard definition of symplectic area as the integral of the symplectic form over any surface dressing the triangle. All appearances of Ω or ω in any equation could be rewritten in terms of trigonometric functions of *symplectic area* $\mathcal{S}$ with label $\eta\kappa_1^2$ (symplectic area goes like the product of lengths along geodesics generated by P_1 and Q_1, whose labels are κ_1 and $\eta\kappa_1$) and *symplectic co-area* s with label $\eta\kappa_2^2$,

$$C_{\eta\kappa_1^2}(2\mathcal{S}) = C_\eta(\Omega) \qquad S_{\eta\kappa_1^2}(2\mathcal{S}) = \frac{S_\eta(\Omega)}{\kappa_1} \qquad C_{\eta\kappa_2^2}(2s) = C_\eta(\omega) \qquad S_{\eta\kappa_2^2}(2s) = \frac{S_\eta(\omega)}{\kappa_2}. \tag{6.47}$$

When $\kappa_1 = 0$, Ω vanishes but $\mathcal{S}$ keeps some finite value, a kind of 'residue' of the generically nonvanishing mixed phase excess Ω. Dually, the same happens for ω and s as $\kappa_2 \to 0$.

6.5. *Dependence and basic equations*

How many independent relations are there in the trigonometric equations for the generic CKD space $\mathbb{C}_\eta S^2_{[\kappa_1],\kappa_2}$? To discuss this, it is better to start with the Cartan sector which includes six phases, to which we will add symplectic area and co-area. For *any* value of the labels κ_1, κ_2, η, there are *four* independent equations between $\phi_i, \Phi_I, \mathcal{S}, s$,

$$\begin{gathered} \phi_i - \Phi_I = \phi_j - \Phi_J = \phi_k - \Phi_K (= \omega - \Omega) \\ (\Omega =)\, \Phi_I + \Phi_J + \phi_k = \kappa_1 2\mathcal{S} \qquad (\omega =)\, \phi_i + \phi_j + \Phi_K = \kappa_2 2s \end{gathered} \tag{6.48}$$

so in any case there are always four such independent Cartan sector quantities. In order to discuss the dependence of the remaining equations, let us consider the determinant of the Gramm matrix whose elements are the 'Hermitian' products of the vectors corresponding to the vertices in the linear ambient space, and its dual quantity:

$$\begin{aligned} \triangle_g &:= 1 - C^2_{\kappa_1}(x_i) - C^2_{\kappa_1}(x_j) - C^2_{\kappa_1}(x_k) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_\eta(\Omega) \\ \triangle_G &:= 1 - C^2_{\kappa_2}(X_I) - C^2_{\kappa_2}(X_J) - C^2_{\kappa_2}(X_K) + 2C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)C_\eta(\omega). \end{aligned} \tag{6.49}$$

From (6.42) and (3.8) it follows that $\triangle_g$ vanishes when $\kappa_1 \to 0$; dually the same happens for $\triangle_G$ when $\kappa_2 \to 0$. Further, the quotient $\triangle_g/\kappa_1^2$ (respectively $\triangle_G/\kappa_2^2$) tends to a well-defined finite limit when $\kappa_1 \to 0$ (respectively $\kappa_2 \to 0$), although still goes to zero when $\kappa_2 \to 0$ (respectively when $\kappa_1 \to 0$). To see this, simplify (6.49) by using (6.19) or (6.20) to obtain

$$\triangle_g = \kappa_1^2\kappa_2 S^2_{\kappa_1}(x_i)S^2_{\kappa_2}(X_J)S^2_{\kappa_1}(x_k) \qquad \triangle_G = \kappa_1\kappa_2^2 S^2_{\kappa_2}(X_I)S^2_{\kappa_1}(x_j)S^2_{\kappa_2}(X_K). \tag{6.50}$$

This suggests introducing two new 'renormalized' quantities γ, Γ in a way similar to (6.46):

$$\gamma := \frac{\triangle_g}{\kappa_1^2} \qquad \Gamma := \frac{\triangle_G}{\kappa_2^2}. \tag{6.51}$$

Relations between Γ, γ and symplectic area and co-area $\mathcal{S}, s$ holding for any $\eta; \kappa_1, \kappa_2$ follow by expressing sines of sides and angles in (6.51) by means of (6.37):

$$\frac{\gamma}{\kappa_2} = S^2_{\kappa_1}(x_i)S^2_{\kappa_2}(X_J)S^2_{\kappa_1}(x_k) = -\frac{S^2_{\eta\kappa_1^2}(2\mathcal{S})S_{\eta\kappa_2^2}(2s)}{S_\eta(\Phi_I)S_\eta(\Phi_J)S_\eta(\Phi_K)} \tag{6.52}$$

$$\frac{\Gamma}{\kappa_1} = S^2_{\kappa_1}(X_I)S^2_{\kappa_2}(x_j)S^2_{\kappa_1}(X_K) = -\frac{S_{\eta\kappa_1^2}(2\mathcal{S})S^2_{\eta\kappa_2^2}(2s)}{S_\eta(\phi_i)S_\eta(\phi_j)S_\eta(\phi_k)}. \tag{6.53}$$

By direct substitution using (6.52) and (6.53) we can also derive the following relations between Γ, γ and the pure sides and angles:

$$\frac{\Gamma}{\kappa_1} = \frac{(\gamma/\kappa_2)^2}{S^2_{\kappa_1}(x_i)S^2_{\kappa_1}(x_j)S^2_{\kappa_1}(x_k)} \qquad \frac{\gamma}{\kappa_2} = \frac{(\Gamma/\kappa_1)^2}{S^2_{\kappa_2}(X_I)S^2_{\kappa_2}(X_J)S^2_{\kappa_2}(X_K)}. \tag{6.54}$$

Let us now discuss the dependence issue in the generic case $\kappa_1 \neq 0, \kappa_2 \neq 0$, where all the equations with tags 2, 3 and 4 follow from 1, *exactly alike* in the real case; a triangle is determined by the sides a, b, c and Ω or by the angles A, B, C and ω. The proofs are also a Hermitian translation of the real ones. For instance, to derive the Hermitian sine theorem from the dual cosine theorems $1I$ in the case $\kappa_2 \neq 0$, start from the identity $S^2_{\kappa_1}(x_i)S^2_{\kappa_2}(X_J) = S^2_{\kappa_1}(x_i)\big(1 - C^2_{\kappa_2}(X_J)\big)\big/\kappa_2$, replace one factor $C_{\kappa_2}(X_J)$ by its expression taken from $1J$, and the other $C_{\kappa_2}(X_J)$ by its *complex conjugate*; then expand

$$S^2_{\kappa_1}(x_i)S^2_{\kappa_2}(X_J) = \frac{1 - C^2_{\kappa_1}(x_i) - C^2_{\kappa_1}(x_j) - C^2_{\kappa_1}(x_k) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_\eta(\Omega)}{\kappa_1^2\kappa_2 S^2_{\kappa_1}(x_k)} \tag{6.55}$$

and use (6.49) and (6.51) to rewrite it as

$$S^2_{\kappa_1}(x_i)S^2_{\kappa_2}(X_J)S^2_{\kappa_1}(x_k) = \frac{\gamma}{\kappa_2}. \tag{6.56}$$

As $\kappa_2 \neq 0$, the rhs of (6.56) is symmetric in the indices ijk (this was also clear from (6.51)). This leads to the sine theorem. Dually, when $\kappa_1 \neq 0$ the sine theorem follows also from the three cosine theorems $1i$. By following the real pattern, the dual Hermitian cosine theorem and equations with tags 3 and 4 can also be derived. Therefore, in the generic $\kappa_1 \neq 0, \kappa_2 \neq 0$ case, either the three ‘complex Hermitian’ cosine theorems $1i$ or $1I$ seen as *six* independent real equations are a set of basic equations. By adding to either choice the four independent ‘Cartan sector’ quantities relating the six phases, $\mathcal{S}, s$, we get a complete set of ten equations relating 14 quantities.

6.5.1. Alternative forms of Hermitian cosine equations when $\kappa_1 = 0$ or $\kappa_2 = 0$. The collapse from (6.13) to (6.42) of the Hermitian cosine equations when $\kappa_1 = 0$ can be circumvented. The imaginary part of (6.13) can be rewritten in terms of the symplectic area $\mathcal{S}$, by using (6.47). For the real part, the procedure mimics the real one [1]: write all cosines of the sides *and* of Ω in terms of versed sines $V_\kappa(x) := (1 - C_\kappa(x))/\kappa$,

$$C_{\kappa_1}(x_i) = 1 - \kappa_1 V_{\kappa_1}(x_i) \qquad C_\eta(\Omega) = C_{\eta\kappa_1^2}(2\mathcal{S}) = 1 - \eta\kappa_1^2 V_{\eta\kappa_1^2}(2\mathcal{S}) \tag{6.57}$$

and then substitute in $1i$, expand, cancel a common factor κ_1 and use the identities for the versed sines of a sum. Thus (6.13) can be rewritten as

$$1\mathrm{i}' \quad \begin{aligned} &V_{\kappa_1}(x_i) - V_{\kappa_1}(x_j + x_k) = S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)\left(C_{\kappa_2}(X_I)C_\eta(\Phi_I) - 1\right) - \eta\kappa_1 V_{\eta\kappa_1^2}(2\mathcal{S})C_{\kappa_1}(x_i) \\ &C_{\kappa_1}(x_i)S_{\eta\kappa_1^2}(2\mathcal{S}) = -S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)S_\eta(\Phi_I) \end{aligned} \tag{6.58}$$

a form meaningful for any value of κ_1 (even if $\kappa_2 = 0, \eta = 0$) reducing when $\kappa_1 = 0$ to

$$\begin{aligned} &x_i^2 = x_j^2 + x_k^2 + 2x_jx_kC_{\kappa_2}(X_I)C_\eta(\Phi_I) \\ &2\mathcal{S} = -x_jx_kC_{\kappa_2}(X_I)S_\eta(\Phi_I). \end{aligned} \tag{6.59}$$

The dual cosine equations $1I$ allow a similar rewriting meaningful for any value of κ_2:

$$1I' \quad \begin{aligned} &V_{\kappa_2}(X_I) - V_{\kappa_2}(X_J + X_K) = S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)\left(C_{\kappa_1}(x_i)C_\eta(\phi_i) - 1\right) \\ &\qquad\qquad - \eta\kappa_2 V_{\eta\kappa_2^2}(2s)C_{\kappa_2}(X_I) \\ &C_{\kappa_2}(X_I)S_{\eta\kappa_2^2}(2s) = -S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)S_\eta(\phi_i). \end{aligned} \tag{6.60}$$

The quantities γ (Γ) can also be given in terms of sides and symplectic area (angles and symplectic co-area) by expressions still meaningful when $\kappa_1 = 0$ ($\kappa_2 = 0$):

$$\begin{aligned}
\gamma &= 2\left(V_{\kappa_1}(x_i)V_{\kappa_1}(x_j) + V_{\kappa_1}(x_j)V_{\kappa_1}(x_k) + V_{\kappa_1}(x_k)V_{\kappa_1}(x_i)\right) - V^2_{\kappa_1}(x_i) - V^2_{\kappa_1}(x_j) - V^2_{\kappa_1}(x_k) \\
&\quad - 2\eta C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)V_{\eta\kappa_1^2}(2\mathcal{S}) - 2\kappa_1 V_{\kappa_1}(x_i)V_{\kappa_1}(x_j)V_{\kappa_1}(x_k) \\
\Gamma &= 2\left(V_{\kappa_2}(X_I)V_{\kappa_2}(X_J) + V_{\kappa_2}(X_J)V_{\kappa_2}(X_K) + V_{\kappa_2}(X_K)V_{\kappa_2}(X_I)\right) - V^2_{\kappa_2}(X_I) \\
&\quad - V^2_{\kappa_2}(X_J) - V^2_{\kappa_2}(X_K) - 2\eta C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)V_{\eta\kappa_2^2}(2s) \\
&\quad - 2\kappa_2 V_{\kappa_2}(X_I)V_{\kappa_2}(X_J)V_{\kappa_2}(X_K).
\end{aligned} \tag{6.61}$$

Thus when $\kappa_1 = 0$, $\kappa_2 \neq 0$, the six real equations $1i'$ are independent and all the remaining equations follow from them and from Cartan sector ones (6.48) just as everything follows from $1i$ when $\kappa_1 \neq 0$. Dually, *mutatis mutandis* from $1I'$ when $\kappa_2 = 0$, $\kappa_1 \neq 0$.

We finally discuss the situation when $\kappa_1 = \kappa_2 = 0$. Lateral and angular phases are equal, $\Phi_I = \phi_i$, and this provides three independent equations. Two further equations are the sine theorem which in this case *cannot* be derived from $1i'$ or $1I'$. This makes *five* independent equations:

$$\Phi_I = \phi_i \qquad \frac{x_i}{X_I} = \frac{x_j}{X_J} = \frac{x_k}{X_K}. \tag{6.62}$$

Further details depend on whether η is zero or not. If $\eta \neq 0$, the equations $1i'/1I'$ read

$$\begin{aligned}
&\mathrm{Re}1i' \quad x_i^2 = x_j^2 + x_k^2 + 2x_jx_kC_\eta(\Phi_I) \qquad && \mathrm{Re}1I' \quad X_I^2 = X_J^2 + X_K^2 + 2X_JX_KC_\eta(\phi_i) \\
&\mathrm{Im}1i' \quad 2\mathcal{S} = -x_jx_kS_\eta(\Phi_I) && \mathrm{Im}1I' \quad 2s = -X_JX_KS_\eta(\phi_i).
\end{aligned} \tag{6.63}$$

Taking into account (6.62), the groups of three equations $\mathrm{Re}1i'$ and $\mathrm{Re}1I'$ are equivalent; either of them can be taken as three further equations and any of these sets imply either relation in the mutually dual pair (whose general form is (6.24) and its dual (6.25)),

$$\frac{x_i}{S_\eta(\Phi_I)} = \frac{x_j}{S_\eta(\Phi_J)} = \frac{x_k}{S_\eta(\Phi_K)} \qquad \frac{X_I}{S_\eta(\phi_i)} = \frac{X_J}{S_\eta(\phi_j)} = \frac{X_K}{S_\eta(\phi_k)} \tag{6.64}$$

from which the three equations in $\mathrm{Im}1i'$ or $\mathrm{Im}1I'$ collapse to a single equation. Taken altogether, these provide another *five* independent equations in (6.63). When $\eta = 0$ the three equations $\mathrm{Re}1i'$ or $\mathrm{Re}1I'$ collapse to a single equation better written as $x_i + x_j + x_k = 0$ or $X_I + X_J + X_K = 0$; these two equations are not independent in view of the sine theorem. In this case, the most contracted forms of (6.24) cannot be derived from previous equations and have to be added as two further independent equations, in either of the two forms,

$$\frac{x_i}{\Phi_I} = \frac{x_j}{\Phi_J} = \frac{x_k}{\Phi_K} \quad \text{or} \quad \frac{X_I}{\phi_i} = \frac{X_J}{\phi_j} = \frac{X_K}{\phi_k} \tag{6.65}$$

and using these equations, each group of three equations $\mathrm{Im}1i'$ or $\mathrm{Im}1I'$ collapses to a single equation. This makes again *five* independent additional equations altogether in

$$\begin{aligned}
&x_i + x_j + x_k = 0 \qquad && \frac{x_i}{\Phi_I} = \frac{x_j}{\Phi_J} = \frac{x_k}{\Phi_K} \qquad && 2\mathcal{S} = -x_jx_k\Phi_I \\
&X_I + X_J + X_K = 0 && \frac{X_I}{\phi_i} = \frac{X_J}{\phi_j} = \frac{X_K}{\phi_k} && 2\mathcal{S} = -X_JX_K\phi_i.
\end{aligned} \tag{6.66}$$

Theorem 3. *The full set of equations of 'complex Hermitian' trigonometry linking the 14 quantities $x_i, X_I, \phi_i, \Phi_I, \mathcal{S}, s$ contains for any value of η, κ_1, κ_2 exactly ten independent equations. Any other equation in the set is a consequence of them. When κ_1 or κ_2 is different*

Table 2. Complex Hermitian sine theorems and relations between symplectic area $\mathcal{S}$, co-area s and mixed phase excesses Ω, ω for the 27 'complex Hermitian' ('CH') CKD spaces. The table is arranged with columns labelled by $\kappa_1 = 1, 0, -1$ and rows by $\kappa_2 = 1, 0, -1$, $(\eta; \kappa_1, \kappa_2)$ are explicitly displayed at each entry. All relations in this table hold the same form regardless of the value of η. The group description of the homogeneous spaces is given in the CKD type notation.

'CH' elliptic $(\eta; +1, +1)$ ${}_\eta SU(3)/{}_\eta U(1) \otimes {}_\eta SU(2)$ $\Omega = 2\mathcal{S}$ $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = \Omega - \omega$ $\omega = 2s$	'CH' Euclidean $(\eta; 0, +1)$ ${}_\eta IU(2)/{}_\eta U(1) \otimes SU(2)$ $\Omega = 0$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = -\omega$ $\omega = 2s$	'CH' hyperbolic $(\eta; -1, +1)$ ${}_\eta SU(2,1)/{}_\eta U(1) \otimes {}_\eta SU(2)$ $\Omega = -2\mathcal{S}$ $\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = \Omega - \omega$ $\omega = 2s$
'CH' co-Euclidean $(\eta; +1, 0)$ 'CH' oscillating NH ${}_\eta IU(2)/{}_\eta U(1) \otimes {}_\eta IU(1)$ $\Omega = 2\mathcal{S}$ $\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = \Omega$ $\omega = 0$	'CH' Galilean $(\eta; 0, 0)$ ${}_\eta IIU(1)/{}_\eta U(1) \otimes {}_\eta IU(1)$ $\Omega = 0$ $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = 0$ $\omega = 0$	'CH' co-Minkowskian $(\eta; -1, 0)$ 'Complex Hermitian' expanding NH ${}_\eta IU(1,1)/{}_\eta U(1) \otimes {}_\eta IU(1)$ $\Omega = -2\mathcal{S}$ $\frac{\sinh a}{A} = \frac{\sinh b}{B} = \frac{\sinh c}{C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = \Omega$ $\omega = 0$
'CH' co-hyperbolic $(\eta; +1, -1)$ 'CH' anti-de Sitter ${}_\eta SU(2,1)/{}_\eta U(1) \otimes {}_\eta SU(1,1)$ $\Omega = 2\mathcal{S}$ $\frac{\sin a}{\sinh A} = \frac{\sin b}{\sinh B} = \frac{\sin c}{\sinh C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = \Omega - \omega$ $\omega = -2s$	'CH' Minkowskian $(\eta; 0, -1)$ ${}_\eta IU(1,1)/{}_\eta U(1) \otimes {}_\eta SU(1,1)$ $\Omega = 0$ $\frac{a}{\sinh A} = \frac{b}{\sinh B} = \frac{c}{\sinh C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = -\omega$ $\omega = -2s$	'CH' doubly hyperbolic $(\eta; -1, -1)$ 'Complex Hermitian' de Sitter ${}_\eta SU(2,1)/{}_\eta U(1) \otimes {}_\eta SU(1,1)$ $\Omega = -2\mathcal{S}$ $\frac{\sinh a}{\sinh A} = \frac{\sinh b}{\sinh B} = \frac{\sinh c}{\sinh C}$ $\Phi_A - \phi_a = \Phi_B - \phi_b$ $= \Phi_C - \phi_c = \Omega - \omega$ $\omega = -2s$

from zero, four such equations are the two phase equations $0ij \equiv 0IJ$ and the two relations $\Omega = \kappa_1 2\mathcal{S}$, $\omega = \kappa_2 2s$. The remaining six independent equations are

- *when $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$, any η, either the equations $1i$ or $1I$;*
- *when $\kappa_1 = 0$ but $\kappa_2 \neq 0$, any η, the equations $1i'$;*
- *when $\kappa_1 \neq 0$ but $\kappa_2 = 0$, any η, the equations $1I'$.*

When both $\kappa_1 = \kappa_2 = 0$, the independent equations are

- *when $\eta \neq 0$, the ten independent equations in (6.62) and (6.63);*
- *when $\eta = 0$, the ten independent equations in (6.62) and (6.66).*

A triangle in any Hermitian spaces in the CKD family is thus characterized by *four* independent quantities, for instance either a, b, c, Ω or A, B, C, ω, which are a dual pair; this was already known for $\mathbb{C}P^2$ or $\mathbb{C}H^2$, but holds for the complete family of 'complex' CKD spaces. Another choice is the six phases linked by the two relations (6.11).

Tables 2, 3, 4 and 5 display the basic equations for the 27 CKD spaces, written in conventional notation. As neither the Cartan sector equations, nor the sine theorem involves the CD label η, these equations are displayed in a single table 2, according to the values of the CK constants (κ_1, κ_2). The remaining basic equations, i.e. Hermitian cosine and dual cosine theorems, which involve the CD label η are given in tables 3, 4 and 5, one table for each value of $\eta = 1, 0, -1$.

Table 3. Complex Hermitian cosine theorems and their duals for the nine *complex Hermitian* CKD spaces ($\eta = 1$). The table is arranged after the values of the pair κ_1, κ_2, and the three labels $(\eta; \kappa_1, \kappa_2)$ are explicitly displayed at each entry. The group description G/H of the homogeneous space is shown only when G has a standard name; this is not the case for $(\kappa_1, \kappa_2) = (0, 0)$. The two spaces of points at the two corners of the last row $\kappa_2 = -1$ are the same, but the corresponding geometries differ by the interchange of first- and second-kind lines generated by either P_1 or P_2. Notice the sign difference in equations involving a, A and involving b, B; c, C and the relevant comments in the main text.

Complex Hermitian elliptic (+1; +1, +1)	Complex Hermitian Euclidean (+1; 0, +1)	Complex Hermitian hyperbolic (+1; −1, +1)
$SU(3)/U(1) \otimes SU(2)$	$IU(2)/U(1) \otimes SU(2)$	$SU(2,1)/U(1) \otimes SU(2)$
$\cos a \cos \Omega = \cos b \cos c - \sin b \sin c \cos A \cos \Psi_A$	$a^2 = b^2 + c^2 + 2bc \cos A \cos \Psi_A$	$\cosh a \cos \Omega = \cosh b \cosh c + \sinh b \sinh c \cos A \cos \Psi_A$
$\cos b \cos \Omega = \cos a \cos c + \sin a \sin c \cos B \cos \Psi_B$	$b^2 = a^2 + c^2 - 2ac \cos B \cos \Psi_B$	$\cosh b \cos \Omega = \cosh a \cosh c - \sinh a \sinh c \cos B \cos \Psi_B$
$\cos c \cos \Omega = \cos a \cos b + \sin a \sin b \cos C \cos \Psi_C$	$c^2 = a^2 + b^2 - 2ab \cos C \cos \Psi_C$	$\cosh c \cos \Omega = \cosh a \cosh b - \sinh a \sinh b \cos C \cos \Psi_C$
$\cos a \sin 2\mathcal{S} = \sin b \sin c \cos A \sin \Psi_A$	$2\mathcal{S} = bc \cos A \sin \Psi_A$	$\cosh a \sin 2\mathcal{S} = \sinh b \sinh c \cos A \sin \Psi_A$
$\cos b \sin 2\mathcal{S} = \sin c \sin a \cos B \sin \Psi_B$	$2\mathcal{S} = ca \cos B \sin \Psi_B$	$\cosh b \sin 2\mathcal{S} = \sinh c \sinh a \cos B \sin \Psi_B$
$\cos c \sin 2\mathcal{S} = \sin a \sin b \cos C \sin \Psi_C$	$2\mathcal{S} = ab \cos C \sin \Psi_C$	$\cosh c \sin 2\mathcal{S} = \sinh a \sinh b \cos C \sin \Psi_C$
$\cos A \sin 2s = \sin B \sin C \cos a \sin \psi_a$	$\cos A \sin 2s = \sin B \sin C \sin \psi_a$	$\cos A \sin 2s = \sin B \sin C \cosh a \sin \psi_a$
$\cos B \sin 2s = \sin C \sin A \cos b \sin \psi_b$	$\cos B \sin 2s = \sin C \sin A \sin \psi_b$	$\cos B \sin 2s = \sin C \sin A \cosh b \sin \psi_b$
$\cos C \sin 2s = \sin A \sin B \cos c \sin \psi_c$	$\cos C \sin 2s = \sin A \sin B \sin \psi_c$	$\cos C \sin 2s = \sin A \sin B \cosh c \sin \psi_c$
$\cos A \cos \omega = \cos B \cos C - \sin B \sin C \cos a \cos \psi_a$	$\cos A \cos \omega = \cos B \cos C - \sin B \sin C \cos \psi_a$	$\cos A \cos \omega = \cos B \cos C - \sin B \sin C \cosh a \cos \psi_a$
$\cos B \cos \omega = \cos A \cos C + \sin A \sin C \cos b \cos \psi_b$	$\cos B \cos \omega = \cos A \cos C + \sin A \sin C \cos \psi_b$	$\cos B \cos \omega = \cos A \cos C + \sin A \sin C \cosh b \cos \psi_b$
$\cos C \cos \omega = \cos A \cos B + \sin A \sin B \cos c \cos \psi_c$	$\cos C \cos \omega = \cos A \cos B + \sin A \sin B \cos \psi_c$	$\cos C \cos \omega = \cos A \cos B + \sin A \sin B \cosh c \cos \psi_c$
Complex Hermitian co-Euclidean (+1; +1, 0)	Complex Hermitian Galilean (+1; 0, 0)	Complex Hermitian co-Minkowskian (+1; −1, 0)
Complex Hermitian oscillating NH		Complex Hermitian expanding NH
$IU(2)/U(1) \otimes IU(1)$		$IU(1,1)/U(1) \otimes IU(1)$
$\cos a \cos \Omega = \cos b \cos c - \sin b \sin c \cos \Psi_A$	$a^2 = b^2 + c^2 + 2bc \cos \Psi_A$	$\cosh a \cos \Omega = \cosh b \cosh c + \sinh b \sinh c \cos \Psi_A$
$\cos b \cos \Omega = \cos a \cos c + \sin a \sin c \cos \Psi_B$	$b^2 = a^2 + c^2 - 2ac \cos \Psi_B$	$\cosh b \cos \Omega = \cosh a \cosh c - \sinh a \sinh c \cos \Psi_B$
$\cos c \cos \Omega = \cos a \cos b + \sin a \sin b \cos \Psi_C$	$c^2 = a^2 + b^2 - 2ab \cos \Psi_C$	$\cosh c \cos \Omega = \cosh a \cosh b - \sinh a \sinh b \cos \Psi_C$
$\cos a \sin 2\mathcal{S} = \sin b \sin c \sin \Psi_A$	$2\mathcal{S} = bc \sin \Psi_A$	$\cosh a \sin 2\mathcal{S} = \sinh b \sinh c \sin \Psi_A$
$\cos b \sin 2\mathcal{S} = \sin c \sin a \sin \Psi_B$	$2\mathcal{S} = ca \sin \Psi_B$	$\cosh b \sin 2\mathcal{S} = \sinh c \sinh a \sin \Psi_B$
$\cos c \sin 2\mathcal{S} = \sin a \sin b \sin \Psi_C$	$2\mathcal{S} = ab \sin \Psi_C$	$\cosh c \sin 2\mathcal{S} = \sinh a \sinh b \sin \Psi_C$
$2s = BC \cos a \sin \psi_a$	$2s = BC \sin \psi_a$	$2s = BC \cosh a \sin \psi_a$
$2s = CA \cos b \sin \psi_b$	$2s = CA \sin \psi_b$	$2s = CA \cosh b \sin \psi_b$
$2s = AB \cos c \sin \psi_c$	$2s = AB \sin \psi_c$	$2s = AB \cosh c \sin \psi_c$
$A^2 = B^2 + C^2 + 2BC \cos a \cos \psi_a$	$A^2 = B^2 + C^2 + 2BC \cos \psi_a$	$A^2 = B^2 + C^2 + 2BC \cosh a \cos \psi_a$
$B^2 = A^2 + C^2 - 2AC \cos b \cos \psi_b$	$B^2 = A^2 + C^2 - 2AC \cos \psi_b$	$B^2 = A^2 + C^2 - 2AC \cosh b \cos \psi_b$
$C^2 = A^2 + B^2 - 2AB \cos c \cos \psi_c$	$C^2 = A^2 + B^2 - 2AB \cos \psi_c$	$C^2 = A^2 + B^2 - 2AB \cosh c \cos \psi_c$

Table 3. (Continued.)

Complex Hermitian co-hyperbolic (+1; +1, −1)	Complex Hermitian Minkowskian (+1; 0, −1)	Complex Hermitian doubly hyperbolic (+1; −1, −1)
Complex Hermitian anti-de Sitter		Complex Hermitian de Sitter
$SU(2,1)/U(1)\otimes SU(1,1)$	$IU(1,1)/U(1)\otimes SU(1,1)$	$SU(2,1)/U(1)\otimes SU(1,1)$
$\cos a\cos\Omega=\cos b\cos c-\sin b\sin c\cosh A\cos\Psi_A$	$a^2=b^2+c^2+2bc\cosh A\cos\Psi_A$	$\cosh a\cos\Omega=\cosh b\cosh c+\sinh b\sinh c\cosh A\cos\Psi_A$
$\cos b\cos\Omega=\cos a\cos c+\sin a\sin c\cosh B\cos\Psi_B$	$b^2=a^2+c^2-2ac\cosh B\cos\Psi_B$	$\cosh b\cos\Omega=\cosh a\cosh c-\sinh a\sinh c\cosh B\cos\Psi_B$
$\cos c\cos\Omega=\cos a\cos b+\sin a\sin b\cosh C\cos\Psi_C$	$c^2=a^2+b^2-2ab\cosh C\cos\Psi_C$	$\cosh c\cos\Omega=\cosh a\cosh b-\sinh a\sinh b\cosh C\cos\Psi_C$
$\cos a\sin 2\mathcal{S}=\sin b\sin c\cosh A\sin\Psi_A$	$2\mathcal{S}=bc\cosh A\sin\Psi_A$	$\cosh a\sin 2\mathcal{S}=\sinh b\sinh c\cosh A\sin\Psi_A$
$\cos b\sin 2\mathcal{S}=\sin c\sin a\cosh B\sin\Psi_B$	$2\mathcal{S}=ca\cosh B\sin\Psi_B$	$\cosh b\sin 2\mathcal{S}=\sinh c\sinh a\cosh B\sin\Psi_B$
$\cos c\sin 2\mathcal{S}=\sin a\sin b\cosh C\sin\Psi_C$	$2\mathcal{S}=ab\cosh C\sin\Psi_C$	$\cosh c\sin 2\mathcal{S}=\sinh a\sinh b\cosh C\sin\Psi_C$
$\cosh A\sin 2s=\sinh B\sinh C\cos a\sin\psi_a$	$\cosh A\sin 2s=\sinh B\sinh C\sin\psi_a$	$\cosh A\sin 2s=\sinh B\sinh C\cosh a\sin\psi_a$
$\cosh B\sin 2s=\sinh C\sinh A\cos b\sin\psi_b$	$\cosh B\sin 2s=\sinh C\sinh A\sin\psi_b$	$\cosh B\sin 2s=\sinh C\sinh A\cosh b\sin\psi_b$
$\cosh C\sin 2s=\sinh A\sinh B\cos c\sin\psi_c$	$\cosh C\sin 2s=\sinh A\sinh B\sin\psi_c$	$\cosh C\sin 2s=\sinh A\sinh B\cosh c\sin\psi_c$
$\cosh A\cos\omega=\cosh B\cosh C+\sinh B\sinh C\cos a\cos\psi_a$	$\cosh A\cos\omega=\cosh B\cosh C+\sinh B\sinh C\cos\psi_a$	$\cosh A\cos\omega=\cosh B\cosh C+\sinh B\sinh C\cosh a\cos\psi_a$
$\cosh B\cos\omega=\cosh A\cosh C-\sinh A\sinh C\cos b\cos\psi_b$	$\cosh B\cos\omega=\cosh A\cosh C-\sinh A\sinh C\cos\psi_b$	$\cosh B\cos\omega=\cosh A\cosh C-\sinh A\sinh C\cosh b\cos\psi_b$
$\cosh C\cos\omega=\cosh A\cosh B-\sinh A\sinh B\cos c\cos\psi_c$	$\cosh C\cos\omega=\cosh A\cosh B-\sinh A\sinh B\cos\psi_c$	$\cosh C\cos\omega=\cosh A\cosh B-\sinh A\sinh B\cosh c\cos\psi_c$

Table 4. 'Complex Hermitian' cosine theorems and their duals for the nine *parabolic complex (dual) 'Hermitian'* CKD spaces ($\eta = 0$). The table is arranged after the values of the pair κ_1, κ_2, and the three labels $(\eta; \kappa_1, \kappa_2)$ are explicitly displayed at each entry. The group description G/H of the homogeneous space is not shown as when $\eta = 0$ the CKD groups are not simple and do not have a standard name. The fiducial role of the trigonometry for the space ($\eta = 0, \kappa_1 = 0, \kappa_2 = 0$) in the centre of this table is clear. All the trigonometries in tables 3, 4 and 5 are deformations of this 'purely linear' one.

'Parabolic complex Hermitian' elliptic $(0; +1, +1)$	'Parabolic complex Hermitian' Euclidean $(0; 0, +1)$	'Parabolic complex Hermitian' hyperbolic $(0; -1, +1)$
$\cos a = \cos b \cos c - \sin b \sin c \cos A$	$a^2 = b^2 + c^2 + 2bc \cos A$	$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A$
$\cos b = \cos a \cos c + \sin a \sin c \cos B$	$b^2 = a^2 + c^2 - 2ac \cos B$	$\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos B$
$\cos c = \cos a \cos b + \sin a \sin b \cos C$	$c^2 = a^2 + b^2 - 2ab \cos C$	$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$
$\cos a 2\mathcal{S} = \sin b \sin c \cos A \,\Psi_A$	$2\mathcal{S} = bc \cos A \,\Psi_A$	$\cosh a 2\mathcal{S} = \sinh b \sinh c \cos A \,\Psi_A$
$\cos b 2\mathcal{S} = \sin c \sin a \cos B \,\Psi_B$	$2\mathcal{S} = ca \cos B \,\Psi_B$	$\cosh b 2\mathcal{S} = \sinh c \sinh a \cos B \,\Psi_B$
$\cos c 2\mathcal{S} = \sin a \sin b \cos C \,\Psi_C$	$2\mathcal{S} = ab \cos C \,\Psi_C$	$\cosh c 2\mathcal{S} = \sinh a \sinh b \cos C \,\Psi_C$
$\cos A 2s = \sin B \sin C \cos a \psi_a$	$\cos A 2s = \sin B \sin C \psi_a$	$\cos A 2s = \sin B \sin C \cosh a \psi_a$
$\cos B 2s = \sin C \sin A \cos b \psi_b$	$\cos B 2s = \sin C \sin A \psi_b$	$\cos B 2s = \sin C \sin A \cosh b \psi_b$
$\cos C 2s = \sin A \sin B \cos c \psi_c$	$\cos C 2s = \sin A \sin B \psi_c$	$\cos C 2s = \sin A \sin B \cosh c \psi_c$
$\cos A = \cos B \cos C - \sin B \sin C \cos a$	$A = B + C$	$\cos A = \cos B \cos C - \sin B \sin C \cosh a$
$\cos B = \cos A \cos C + \sin A \sin C \cos b$	$B = A - C$	$\cos B = \cos A \cos C + \sin A \sin C \cosh b$
$\cos C = \cos A \cos B + \sin A \sin B \cos c$	$C = A - B$	$\cos C = \cos A \cos B + \sin A \sin B \cosh c$
'Parabolic complex Hermitian' co-Euclidean $(0; +1, 0)$	'Parabolic complex Hermitian' Galilean $(0; 0, 0)$	'Parabolic complex Hermitian' co-Minkowskian $(0; -1, 0)$
'Parabolic complex Hermitian' oscillating NH		'Parabolic complex Hermitian' expanding NH
$a = b + c$	$a = b + c$	$a = b + c$
$b = a - c$	$b = a - c$	$b = a - c$
$c = a - b$	$c = a - b$	$c = a - b$
$\cos a 2\mathcal{S} = \sin b \sin c \,\Psi_A$	$2\mathcal{S} = bc \,\Psi_A$	$\cosh a 2\mathcal{S} = \sinh b \sinh c \,\Psi_A$
$\cos b 2\mathcal{S} = \sin c \sin a \,\Psi_B$	$2\mathcal{S} = ca \,\Psi_B$	$\cosh b 2\mathcal{S} = \sinh c \sinh a \,\Psi_B$
$\cos c 2\mathcal{S} = \sin a \sin b \,\Psi_C$	$2\mathcal{S} = ab \,\Psi_C$	$\cosh c 2\mathcal{S} = \sinh a \sinh b \,\Psi_C$
$2s = BC \cos a \psi_a$	$2s = BC \psi_a$	$2s = BC \cosh a \psi_a$
$2s = CA \cos b \psi_b$	$2s = CA \psi_b$	$2s = CA \cosh b \psi_b$
$2s = AB \cos c \psi_c$	$2s = AB \psi_c$	$2s = AB \cosh c \psi_c$
$A^2 = B^2 + C^2 + 2BC \cos a$	$A = B + C$	$A^2 = B^2 + C^2 + 2BC \cosh a$
$B^2 = A^2 + C^2 - 2AC \cos b$	$B = A - C$	$B^2 = A^2 + C^2 - 2AC \cosh b$
$C^2 = A^2 + B^2 - 2AB \cos c$	$C = A - B$	$C^2 = A^2 + B^2 - 2AB \cosh c$

Table 4. (Continued.)

'Parabolic complex Hermitian' co-hyperbolic (0; +1, −1) 'Parabolic complex Hermitian' anti-de Sitter	'Parabolic complex Hermitian' Minkowskian (0; 0, −1)	'Parabolic complex Hermitian' doubly hyperbolic (0; −1, −1) 'Parabolic complex Hermitian' de Sitter
$\cos a = \cos b \cos c - \sin b \sin c \cosh A$	$a^2 = b^2 + c^2 + 2bc \cosh A$	$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cosh A$
$\cos b = \cos a \cos c + \sin a \sin c \cosh B$	$b^2 = a^2 + c^2 - 2ac \cosh B$	$\cosh b = \cosh a \cosh c - \sinh a \sinh c \cosh B$
$\cos c = \cos a \cos b + \sin a \sin b \cosh C$	$c^2 = a^2 + b^2 - 2ab \cosh C$	$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh C$
$\cos a 2\mathcal{S} = \sin b \sin c \cosh A\ \Psi_A$	$2\mathcal{S} = bc \cosh A\ \Psi_A$	$\cosh a 2\mathcal{S} = \sinh b \sinh c \cosh A\ \Psi_A$
$\cos b 2\mathcal{S} = \sin c \sin a \cosh B\ \Psi_B$	$2\mathcal{S} = ca \cosh B\ \Psi_B$	$\cosh b 2\mathcal{S} = \sinh c \sinh a \cosh B\ \Psi_B$
$\cos c 2\mathcal{S} = \sin a \sin b \cosh C\ \Psi_C$	$2\mathcal{S} = ab \cosh C\ \Psi_C$	$\cosh c 2\mathcal{S} = \sinh a \sinh b \cosh C\ \Psi_C$
$\cosh A 2s = \sinh B \sinh C \cos a \psi_a$	$\cosh A 2s = \sinh B \sinh C \psi_a$	$\cosh A 2s = \sinh B \sinh C \cosh a \psi_a$
$\cosh B 2s = \sinh C \sinh A \cos b \psi_b$	$\cosh B 2s = \sinh C \sinh A \psi_b$	$\cosh B 2s = \sinh C \sinh A \cosh b \psi_b$
$\cosh C 2s = \sinh A \sinh B \cos c \psi_c$	$\cosh C 2s = \sinh A \sinh B \psi_c$	$\cosh C 2s = \sinh A \sinh B \cosh c \psi_c$
$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cos a$	$A = B + C$	$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cosh a$
$\cosh B = \cosh A \cosh C - \sinh A \sinh C \cos b$	$B = A - C$	$\cosh B = \cosh A \cosh C - \sinh A \sinh C \cosh b$
$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos c$	$C = A - B$	$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cosh c$

Table 5. 'Complex Hermitian' cosine theorems and their duals for the nine *split complex 'Hermitian'* CKD spaces ($\eta = -1$). The table is arranged after the values κ_1, κ_2, and the labels $(\eta; \kappa_1, \kappa_2)$ are explicitly displayed at each entry. The group description G/H of the homogeneous space is shown only when G has a standard name. The spaces at the four corners are equal, but the trigonometric equations in these geometries are different as they correspond to triangles with geodesic sides of the four not conjugate different possible types.

'Split complex Hermitian' elliptic $(-1; +1, +1)$	'Split complex Hermitian' Euclidean $(-1; 0, +1)$	'Split complex Hermitian' hyperbolic $(-1; -1, +1)$
$SL(3,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R})$	$IGL(2,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R})$	$SL(3,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R})$
$\cos a \cosh\Omega = \cos b\cos c - \sin b\sin c\cos A\cosh\Psi_A$	$a^2 = b^2 + c^2 + 2bc\cos A\cosh\Psi_A$	$\cosh a\cosh\Omega = \cosh b\cosh c + \sinh b\sinh c\cos A\cosh\Psi_A$
$\cos b\cosh\Omega = \cos a\cos c + \sin a\sin c\cos B\cosh\Psi_B$	$b^2 = a^2 + c^2 - 2ac\cos B\cosh\Psi_B$	$\cosh b\cosh\Omega = \cosh a\cosh c - \sinh a\sinh c\cos B\cosh\Psi_B$
$\cos c\cosh\Omega = \cos a\cos b + \sin a\sin b\cos C\cosh\Psi_C$	$c^2 = a^2 + b^2 - 2ab\cos C\cosh\Psi_C$	$\cosh c\cosh\Omega = \cosh a\cosh b - \sinh a\sinh b\cos C\cos\Psi_C$
$\cos a\sinh 2\mathcal{S} = \sin b\sin c\cos A\sinh\Psi_A$	$2\mathcal{S} = bc\cos A\sinh\Psi_A$	$\cosh a\sinh 2\mathcal{S} = \sinh b\sinh c\cos A\sinh\Psi_A$
$\cos b\sinh 2\mathcal{S} = \sin c\sin a\cos B\sinh\Psi_B$	$2\mathcal{S} = ca\cos B\sinh\Psi_B$	$\cosh b\sinh 2\mathcal{S} = \sinh c\sinh a\cos B\sinh\Psi_B$
$\cos c\sinh 2\mathcal{S} = \sin a\sin b\cos C\sinh\Psi_C$	$2\mathcal{S} = ab\cos C\sinh\Psi_C$	$\cosh c\sinh 2\mathcal{S} = \sinh a\sinh b\cos C\sinh\Psi_C$
$\cos A\sinh 2s = \sin B\sin C\cos a\sinh\psi_a$	$\cos A\sinh 2s = \sin B\sin C\sinh\psi_a$	$\cos A\sinh 2s = \sin B\sin C\cosh a\sinh\psi_a$
$\cos B\sinh 2s = \sin C\sin A\cos b\sinh\psi_b$	$\cos B\sinh 2s = \sin C\sin A\sinh\psi_b$	$\cos B\sinh 2s = \sin C\sin A\cosh b\sinh\psi_b$
$\cos C\sinh 2s = \sin A\sin B\cos c\sinh\psi_c$	$\cos C\sinh 2s = \sin A\sin B\sinh\psi_c$	$\cos C\sinh 2s = \sin A\sin B\cosh c\sinh\psi_c$
$\cos A\cos\omega = \cos B\cos C - \sin B\sin C\cos a\cosh\psi_a$	$\cos A\cos\omega = \cos B\cos C - \sin B\sin C\cosh\psi_a$	$\cos A\cos\omega = \cos B\cos C - \sin B\sin C\cosh a\cosh\psi_a$
$\cos B\cos\omega = \cos A\cos C + \sin A\sin C\cos b\cosh\psi_b$	$\cos B\cos\omega = \cos A\cos C + \sin A\sin C\cosh\psi_b$	$\cos B\cos\omega = \cos A\cos C + \sin A\sin C\cosh b\cosh\psi_b$
$\cos C\cos\omega = \cos A\cos B + \sin A\sin B\cos c\cosh\psi_c$	$\cos C\cos\omega = \cos A\cos B + \sin A\sin B\cosh\psi_c$	$\cos C\cos\omega = \cos A\cos B + \sin A\sin B\cosh c\cosh\psi_c$
'Split complex Hermitian' co-Euclidean $(-1; +1, 0)$	'Split complex Hermitian' Galilean $(-1; 0, 0)$	'Split complex Hermitian' co-Minkowskian $(-1; -1, 0)$
'Split complex Hermitian' oscillating NH		'Split complex Hermitian' expanding NH
$IGL(2,\mathbb{R})/SO(1,1)\otimes IGL(1,\mathbb{R})$		$IGL(2,\mathbb{R})/SO(1,1)\otimes IGL(1,\mathbb{R})$
$\cos a\cosh\Omega = \cos b\cos c - \sin b\sin c\cosh\Psi_A$	$a^2 = b^2 + c^2 + 2bc\cosh\Psi_A$	$\cosh a\cosh\Omega = \cosh b\cosh c + \sinh b\sinh c\cosh\Psi_A$
$\cos b\cosh\Omega = \cos a\cos c + \sin a\sin c\cosh\Psi_B$	$b^2 = a^2 + c^2 - 2ac\cosh\Psi_B$	$\cosh b\cosh\Omega = \cosh a\cosh c - \sinh a\sinh c\cosh\Psi_B$
$\cos c\cosh\Omega = \cos a\cos b + \sin a\sin b\cosh\Psi_C$	$c^2 = a^2 + b^2 - 2ab\cosh\Psi_C$	$\cosh c\cosh\Omega = \cosh a\cosh b - \sinh a\sinh b\cos\Psi_C$
$\cos a\sinh 2\mathcal{S} = \sin b\sin c\sinh\Psi_A$	$2\mathcal{S} = bc\sinh\Psi_A$	$\cosh a\sinh 2\mathcal{S} = \sinh b\sinh c\sinh\Psi_A$
$\cos b\sinh 2\mathcal{S} = \sin c\sin a\sinh\Psi_B$	$2\mathcal{S} = ca\sinh\Psi_B$	$\cosh b\sinh 2\mathcal{S} = \sinh c\sinh a\sinh\Psi_B$
$\cos c\sinh 2\mathcal{S} = \sin a\sin b\sinh\Psi_C$	$2\mathcal{S} = ab\sinh\Psi_C$	$\cosh c\sinh 2\mathcal{S} = \sinh a\sinh b\sinh\Psi_C$
$2s = BC\cos a\sinh\psi_a$	$2s = BC\sinh\psi_a$	$2s = BC\cosh a\sinh\psi_a$
$2s = CA\cos b\sinh\psi_b$	$2s = CA\sinh\psi_b$	$2s = CA\cosh b\sinh\psi_b$
$2s = AB\cos c\sinh\psi_c$	$2s = AB\sinh\psi_c$	$2s = AB\cosh c\sinh\psi_c$
$A^2 = B^2 + C^2 + 2BC\cos a\cosh\psi_a$	$A^2 = B^2 + C^2 + 2BC\cosh\psi_a$	$A^2 = B^2 + C^2 + 2BC\cosh a\cosh\psi_a$
$B^2 = A^2 + C^2 - 2AC\cos b\cosh\psi_b$	$B^2 = A^2 + C^2 - 2AC\cosh\psi_b$	$B^2 = A^2 + C^2 - 2AC\cosh b\cosh\psi_b$
$C^2 = A^2 + B^2 - 2AB\cos c\cosh\psi_c$	$C^2 = A^2 + B^2 - 2AB\cosh\psi_c$	$C^2 = A^2 + B^2 - 2AB\cosh c\cosh\psi_c$

Table 5. (Continued.)

'Split complex Hermitian' co-hyperbolic $(-1; +1, -1)$	'Split complex Hermitian' Minkowskian $(-1; 0, -1)$	'Split complex Hermitian' doubly hyperbolic $(-1; -1, -1)$
'split complex Hermitian' anti-de Sitter		'split complex Hermitian' de Sitter
$SL(3,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R})$	$IGL(2,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R})$	$SL(3,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R})$
$\cos a \cosh\Omega = \cos b \cos c - \sin b \sin c \cosh A \cosh\Psi_A$	$a^2 = b^2 + c^2 + 2bc \cosh A \cosh\Psi_A$	$\cosh a \cosh\Omega = \cosh b \cosh c + \sinh b \sinh c \cosh A \cosh\Psi_A$
$\cos b \cosh\Omega = \cos a \cos c + \sin a \sin c \cosh B \cosh\Psi_B$	$b^2 = a^2 + c^2 - 2ac \cosh B \cosh\Psi_B$	$\cosh b \cosh\Omega = \cosh a \cosh c - \sinh a \sinh c \cosh B \cosh\Psi_B$
$\cos c \cosh\Omega = \cos a \cos b + \sin a \sin b \cosh C \cosh\Psi_C$	$c^2 = a^2 + b^2 - 2ab \cosh C \cosh\Psi_C$	$\cosh c \cosh\Omega = \cosh a \cosh b - \sinh a \sinh b \cosh C \cos\Psi_C$
$\cos a \sinh 2\mathcal{S} = \sin b \sin c \cosh A \sinh\Psi_A$	$2\mathcal{S} = bc \cosh A \sinh\Psi_A$	$\cosh a \sinh 2\mathcal{S} = \sinh b \sinh c \cosh A \sinh\Psi_A$
$\cos b \sinh 2\mathcal{S} = \sin c \sin a \cosh B \sinh\Psi_B$	$2\mathcal{S} = ca \cosh B \sinh\Psi_B$	$\cosh b \sinh 2\mathcal{S} = \sinh c \sinh a \cosh B \sinh\Psi_B$
$\cos c \sinh 2\mathcal{S} = \sin a \sin b \cosh C \sinh\Psi_C$	$2\mathcal{S} = ab \cosh C \sinh\Psi_C$	$\cosh c \sinh 2\mathcal{S} = \sinh a \sinh b \cosh C \sinh\Psi_C$
$\cosh A \sinh 2s = \sinh B \sinh C \cos a \sinh\psi_a$	$\cosh A \sinh 2s = \sinh B \sinh C \sinh\psi_a$	$\cosh A \sinh 2s = \sinh B \sinh C \cosh a \sinh\psi_a$
$\cosh B \sinh 2s = \sinh C \sinh A \cos b \sinh\psi_b$	$\cosh B \sinh 2s = \sinh C \sinh A \sinh\psi_b$	$\cosh B \sinh 2s = \sinh C \sinh A \cosh b \sinh\psi_b$
$\cosh C \sinh 2s = \sinh A \sinh B \cos c \sinh\psi_c$	$\cosh C \sinh 2s = \sinh A \sinh B \sinh\psi_c$	$\cosh C \sinh 2s = \sinh A \sinh B \cosh c \sinh\psi_c$
$\cosh A \cos\omega = \cosh B \cosh C + \sinh B \sinh C \cos a \cosh\psi_a$	$\cosh A \cos\omega = \cosh B \cosh C + \sinh B \sinh C \cosh\psi_a$	$\cosh A \cos\omega = \cosh B \cosh C + \sinh B \sinh C \cosh a \cosh\psi_a$
$\cosh B \cos\omega = \cosh A \cosh C - \sinh A \sinh C \cos b \cosh\psi_b$	$\cosh B \cos\omega = \cosh A \cosh C - \sinh A \sinh C \cosh\psi_b$	$\cosh B \cos\omega = \cosh A \cosh C - \sinh A \sinh C \cosh b \cosh\psi_b$
$\cosh C \cos\omega = \cosh A \cosh B - \sinh A \sinh B \cos c \cosh\psi_c$	$\cosh C \cos\omega = \cosh A \cosh B - \sinh A \sinh B \cosh\psi_c$	$\cosh C \cos\omega = \cosh A \cosh B - \sinh A \sinh B \cosh c \cosh\psi_c$

6.6. Symmetric invariants and existence conditions

Several Hermitian trigonometric equations have a structure similar to the sine theorem: a 'one-element' expression involving only one index (vertex, opposite side) has the same value for the two remaining ones:

$$\Phi_I - \phi_i = \Omega - \omega \qquad \frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} =: \tau \qquad \frac{S_{\kappa_1}(2x_i)}{S_\eta(\Phi_I)C_{\kappa_2}(X_I)} =: \xi \qquad \frac{S_{\kappa_2}(2X_I)}{S_\eta(\phi_i)C_{\kappa_1}(x_i)} =: \Xi. \tag{6.67}$$

Under duality $\tau \leftrightarrow \frac{1}{\tau}$ and $\xi \leftrightarrow \Xi$. Other such 'one-element'-type equations have values which can be expressed in terms of the three triangle invariants τ, ξ, Ξ:

$$\frac{S_{\kappa_1}(x_i)S_{\kappa_2}(X_I)}{S_\eta(\phi_i)S_\eta(\Phi_I)} = \frac{1}{4}\xi\,\Xi \qquad \frac{C_{\kappa_1}(x_i)T_{\kappa_2}(X_I)}{S_\eta(\Phi_I)} = \frac{1}{2}\frac{\xi}{\tau} \qquad \frac{C_{\kappa_2}(X_I)T_{\kappa_1}(x_i)}{S_\eta(\phi_i)} = \frac{1}{2}\Xi\tau. \tag{6.68}$$

A second type is alike formulae allowing the introduction of Ω, ω; a 'cyclic' expression invariant under any cyclic permutation of the three indices involved

$$\begin{array}{ll} \Phi_I + \Phi_J + \phi_k =: \Omega = \kappa_1 2\mathcal{S} & \phi_i + \phi_j + \Phi_K =: \omega = \kappa_2 2s \\ S_{\kappa_2}(X_I)S_{\kappa_2}(X_J)S_{\kappa_1}(x_k) = \sqrt{\Gamma/\kappa_1} & S_{\kappa_1}(x_i)S_{\kappa_1}(x_j)S_{\kappa_2}(X_K) = \sqrt{\gamma/\kappa_2}. \end{array} \tag{6.69}$$

There is *no* essential difference between the 'one-element' and 'cyclic' types; it turns out to be possible to express the 'one-element' invariants in an explicitly 'cyclic' form:

$$\begin{aligned}
&\frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} =: \tau = \frac{S_{\kappa_1}(x_i)S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)}{\sqrt{\gamma/\kappa_2}} = \frac{\sqrt{\gamma/\kappa_2}}{\sqrt{\Gamma/\kappa_1}} \\
&\frac{S_{\kappa_2}(X_I)}{S_{\kappa_1}(x_i)} =: \frac{1}{\tau} = \frac{S_{\kappa_2}(X_I)S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)}{\sqrt{\Gamma/\kappa_1}} = \frac{\sqrt{\Gamma/\kappa_1}}{\sqrt{\gamma/\kappa_2}} \\
&\frac{S_{\kappa_1}(2x_i)}{S_\eta(\Phi_I)C_{\kappa_2}(X_I)} =: \xi = -2\frac{S_{\kappa_1}(x_i)S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)}{S_{\eta\kappa_1^2}(2\mathcal{S})} = -2\frac{\gamma/\kappa_2}{\sqrt{\Gamma/\kappa_1}}\frac{1}{S^2_{\eta\kappa_1^2}(2\mathcal{S})} \\
&\frac{S_{\kappa_2}(2X_I)}{S_\eta(\phi_i)C_{\kappa_1}(x_i)} =: \Xi = -2\frac{S_{\kappa_2}(X_I)S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)}{S_{\eta\kappa_2^2}(2s)} = -2\frac{\Gamma/\kappa_1}{\sqrt{\gamma/\kappa_2}}\frac{1}{S^2_{\eta\kappa_2^2}(2s)}.
\end{aligned} \tag{6.70}$$

The two quantities $\sqrt{\gamma/\kappa_2}$ and $\sqrt{\Gamma/\kappa_1}$ must be real for the triangle to exist. Therefore, any triangle must satisfy the inequalities,

$$\gamma/\kappa_2 \geqslant 0 \qquad \Gamma/\kappa_1 \geqslant 0 \tag{6.71}$$

which apply to *any* member of the CKD family of 'complex Hermitian' spaces, notwithstanding any restriction for the sides and angles. Brehm's inequalities under which a triangle with prescribed values for sides and shape invariant exists in $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are simply the transcription of the condition $\frac{\gamma}{\kappa_2} \geqslant 0$ to the complex CK spaces with $\eta = 1$; $\kappa_1 \neq 0, \kappa_2 = 1$. Thus $\gamma \geqslant 0$ or equivalently $\triangle_g = \kappa_1^2\gamma \geqslant 0$; by using (6.49) this gives an inequality covering simultaneously those given by Brehm for $\mathbb{C}P^N$ ($\kappa_1 > 0$) and $\mathbb{C}H^N$ ($\kappa_1 < 0$); note that our Ω refers to Brehm's ω as:

$$\triangle_g = 1 - C^2_{\kappa_1}(x_i) - C^2_{\kappa_1}(x_j) - C^2_{\kappa_1}(x_k) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)\cos\Omega \geqslant 0. \tag{6.72}$$

It is also worth highlighting the translation of inequalities (6.71) by using (6.35); this brings them in terms of angular and lateral phases, and symplectic area and co-area,

$$-\frac{S_{\eta\kappa_2^2}(2s)}{S_\eta(\Phi_I)S_\eta(\Phi_J)S_\eta(\Phi_K)} \geqslant 0 \qquad -\frac{S_{\eta\kappa_1^2}(2\mathcal{S})}{S_\eta(\phi_i)S_\eta(\phi_j)S_\eta(\phi_k)} \geqslant 0. \tag{6.73}$$

The same translation can be done in expressions (6.70), thereby expressing the values τ, ξ, Ξ in terms of lateral and angular phases, symplectic area and co-area,

$$\frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} =: \tau = \sqrt{\frac{S_\eta(\phi_i)S_\eta(\phi_j)S_\eta(\phi_k)S_{\eta\kappa_1^2}(2\mathcal{S})}{S_\eta(\Phi_I)S_\eta(\Phi_J)S_\eta(\Phi_K)S_{\eta\kappa_2^2}(2s)}}$$
$$\frac{S_{\kappa_1}(2x_i)}{S_\eta(\Phi_I)C_{\kappa_2}(X_I)} =: \xi = \sqrt{\frac{-4S_\eta(\phi_i)S_\eta(\phi_j)S_\eta(\phi_k)S_{\eta\kappa_1^2}(2\mathcal{S})}{S^2_\eta(\Phi_I)S^2_\eta(\Phi_J)S^2_\eta(\Phi_K)}} = 2\tau\sqrt{\frac{-S_{\eta\kappa_2^2}(2s)}{S_\eta(\Phi_I)S_\eta(\Phi_J)S_\eta(\Phi_K)}}$$
$$\frac{S_{\kappa_2}(2X_I)}{S_\eta(\phi_i)C_{\kappa_1}(x_i)} =: \Xi = \sqrt{\frac{-4S_\eta(\Phi_I)S_\eta(\Phi_J)S_\eta(\Phi_K)S_{\eta\kappa_2^2}(2s)}{S^2_\eta(\phi_i)S^2_\eta(\phi_j)S^2_\eta(\phi_k)}} = \frac{2}{\tau}\sqrt{\frac{-S_{\eta\kappa_1^2}(2\mathcal{S})}{S_\eta(\phi_i)S_\eta(\phi_j)S_\eta(\phi_k)}} \tag{6.74}$$

in whose form the existence inequalities (6.73) are evident. Inequalities (6.71) are analogous to the existence conditions $\frac{E}{\kappa_1} \leqslant 0$, $\frac{e}{\kappa_2} \leqslant 0$ for the half-excesses $E = \Delta/2$, $e = \delta/2$ appearing in the trigonometry of real spaces. Indeed both can be made to follow from inequalities (6.71) as applied to the determinant of the Gramm matrices whose elements are either the symmetric or 'Hermitian' scalar products of vertices or of poles of the sides; in the real case these inequalities can be simplified to $\frac{E}{\kappa_1} \leqslant 0$, $\frac{e}{\kappa_2} \leqslant 0$ while this is not possible in the 'Hermitian' case, where the inequalities stay in the form (6.71).

6.7. The three special cases: collinear triangles, concurrent triangles and purely real triangles

6.7.1. Complex collinear triangles. Many equations obtained can be stated in two similar versions, one involving sides (angles) and the other involving twice the sides (angles): examples of such pairs are (6.15), (6.16) or (6.12), (6.21) and their duals. This corresponds to two special non-generic types of triangles, whose trigonometry reduces to a (known) simpler form. The first special case corresponds to a triangle determined by three 'complex' collinear vertices, hence collapsing from the 'complex' 2D CK Hermitian space ${}_\eta SU_{\kappa_1,\kappa_2}(3)/({}_\eta U(1) \otimes {}_\eta SU_{\kappa_2}(2))$ to a 'complex' 1D subspace ${}_\eta SU_{\kappa_1}(2)/{}_\eta U(1)$. Depending on whether $\kappa_1 > 0, = 0, < 0$, this space is the elliptic, Euclidean or hyperbolic Hermitian 'complex' line. Sides are all different from zero, but angles X_I, X_J, X_K must be zero (or straight), and thus satisfy $S_{\kappa_2}(X_I) = S_{\kappa_2}(X_J) = S_{\kappa_2}(X_K) = 0$ (hence $\gamma = \Gamma = 0$). Equations $1J$ reduce to $e^{i\omega} = 1$, thus $\omega = 0$ and from $0iI$ we get

$$\omega = 0 \qquad \Phi_I - \phi_i = \Omega. \tag{6.75}$$

The SR double cosine for sides (6.16) and double sine for sides (6.21) become

$$C_{\kappa_1}(2x_j) = C_{\kappa_1}(2x_i)C_{\kappa_1}(2x_k) - \kappa_1 S_{\kappa_1}(2x_i)S_{\kappa_1}(2x_k)C_\eta(\Phi_J). \tag{6.76}$$

$$\frac{S_{\kappa_1}(2x_j)}{S_\eta(\Phi_J)} = \xi. \tag{6.77}$$

As $S_{\kappa_1}(2x_j) = 2S_{4\kappa_1}(x_j)$, $C_{\kappa_1}(2x_j) = C_{4\kappa_1}(x_j)$, *all* Hermitian trigonometric equations reduce in this case to the trigonometry of a triangle with sides x_i, x_j, x_k and angles Φ_I, Φ_J, Φ_K in a real CK space with labels $4\kappa_1$ for sides and η for angles, for which (6.76), (6.77) are the real cosine and sine theorems. This corresponds to the fact that the submanifold ${}_\eta SU_{\kappa_1}(2)/{}_\eta U(1)$ has constant curvature $4\kappa_1$, which is equal to the constant holomorphic curvature of the 'complex' 2D CK Hermitian space; this is the sectional curvature of ${}_\eta SU_{\kappa_1,\kappa_2}(3)/({}_\eta U(1) \otimes {}_\eta SU_{\kappa_2}(2))$ along the 1D 'complex' subspaces of the tangent space.

By using (6.41), and recalling $\omega = 0$, the auxiliar triangle angular excess $\Phi_I + \Phi_J + \Phi_K$ turns out to be equal to 2Ω, and thus Ω plays the role of the angular half-excess denoted by E in the previous paper on real-type trigonometry [1]. It is worth remarking that the area $\mathcal{A}$ of this auxiliar triangle is related to its angular excess as $\mathcal{A} = \frac{2\Omega}{4\kappa_1} = \frac{\Omega}{2\kappa_1}$, thus coinciding with the original triangle symplectic area $\mathcal{S}$. The lateral phases ϕ_i turn out to coincide with the three auxiliar angles denoted by E_I in [1]. In terms of the symmetric invariants, the 'collinear' case corresponds to

$$\begin{array}{llllll} \Delta = 0 & \Delta_\Phi = 2\Omega & \Omega = \Omega & \gamma = 0 & \mathcal{S} = \mathcal{S} & \xi = \xi \\ \delta = \delta & \delta_\phi = -\Omega & \omega = 0 & \Gamma = 0 & s = 0 & \Xi = 0 \end{array} \qquad \tau = \infty. \tag{6.78}$$

6.7.2. Concurrent triangles. The second special case, dual to the previous one, corresponds to a triangle determined by three geodesic sides concurrent at the same point. Then sines of sides vanish: $S_{\kappa_1}(x_i) = S_{\kappa_1}(x_j) = S_{\kappa_1}(x_k) = 0$. Here $\Omega = 0$, and the SR dual double cosine (6.18) and SR dual double sine (6.22) become the cosine and sine theorems for a triangle with sides X_I, X_J, X_K and angles ϕ_i, ϕ_j, ϕ_k in a real CK space with labels $4\kappa_2$ for sides and η for angles. For the angular excess of this auxiliar triangle, we have $\phi_i + \phi_j + \phi_k = 2\omega$, and thus ω plays the role of the angular half-excess (E in [1]). In terms of the symmetric invariants, this case correspond to

$$\begin{array}{llllll} \Delta = \Delta & \Delta_\Phi = -\omega & \Omega = 0 & \gamma = 0 & \mathcal{S} = 0 & \xi = 0 \\ \delta = 0 & \delta_\phi = 2\omega & \omega = \omega & \Gamma = 0 & s = s & \Xi = \Xi \end{array} \qquad \tau = 0. \tag{6.79}$$

6.7.3. Totally real triangles. The third special case corresponds to a triangle for which the lateral and angular phase factors $e^{i\phi_i}$ and $e^{i\Phi_I}$ are *real*, and sides and angles are different from zero. This *totally real* triangle is contained in a totally real and totally geodesic submanifold, isometric to $SO_{\kappa_1,\kappa_2}(3)/(O(1)\otimes SO_{\kappa_2}(2))$, and locally isometric (as $O(1) \equiv Z_2$) to $SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$; for $\kappa_1 = 1, \kappa_2 = 1$ this is the real projective space $\mathbb{R}P^2$. Sines of either set of phases vanish whenever the sines of the other set vanish (see (6.27)). Angular and lateral phase excesses, Ω and ω, and symplectic area and co-area have vanishing sines. Each individual phase ϕ_i or Φ_I can thus have only two values, either 0 or twice a quadrant of label η, which have opposite cosines ± 1. In terms of the symmetric invariants, this case corresponds to the values

$$\begin{array}{llllll} \Delta = \Delta & \Delta_\Phi = 0 & \Omega = 0 & \gamma = \gamma & \mathcal{S} = 0 & \xi = \infty \\ \delta = \delta & \delta_\phi = 0 & \omega = 0 & \Gamma = \Gamma & s = 0 & \Xi = \infty \end{array} \qquad \tau = \tau. \tag{6.80}$$

This reduction also provides an approach to the trigonometry of *real* projective planes, requiring as triangle elements, in addition to sides and angles, a set of *discrete* phases, entering the equations only through their cosines $\varepsilon_i = C_\eta(\phi_i) = \pm 1$; $\varepsilon_I = C_\eta(\Phi_I) = \pm 1$. Thus Hermitian trigonometry of the 'complex' spaces simultaneously affords, if we restrict phases to these two possible discrete values, the trigonometry [21] of the real 'projective' CK space family (to which $\mathbb{R}P^2$ belongs).

7. Overview of physical applications

There is actually a strong link between this study and physics: the mathematical structure underlying the quantum state space belongs to the 'complex Hermitian' CKD family. So as a by-product of this work we obtain the basic equations of the *‘trigonometry of the quantum state space’* [27, 28]. Any Hilbert space appears in the CKD family as a Hermitian *Euclidean* space (thus with labels $\eta = 1; \kappa_1 = 0, \kappa_2 = \kappa_3 = \cdots = 1$) and its *projective* Hilbert space (the quantum space of states) appears as the Hermitian elliptic space ($\eta = 1; \kappa_1 = \kappa > 0, \kappa_2 = \kappa_3 = \cdots = 1$). Geometric phases are related to trigonometric quantities: for the simplest ‘triangle type’ loop in the quantum state space, the Anandan–Aharonov phase appears intriguingly as one of the triangle invariants introduced by Blaschke and Terheggen 60 years ago. The paper by Sudarshan, Ananadan and Govindarajan [29] gives a group theoretical derivation of the Anandan–Aharonov phase (equal to triangle symplectic area) for an infinitesimal triangle loop in $\mathbb{C}P^N$; this result appears as a particular case of our *exact* expressions linking triangle elements for any *finite* triangle. The role of symplectic area for geodesic triangles in connection with coherent states and geometric phases has also been recently discussed by Berceanu [30] and Boya, Perelomov and Santander [31]. A separate, more physically oriented paper [32] will be devoted to the trigonometry of the quantum space of states, in relation to geometric phases and, in general, with a view towards a more geometrical formulation of quantum mechanics [33].

The identification of the quantum space of states as a member of this complete CKD family of spaces also makes it natural to inquire about whether or not the labels η, κ_1, κ_2 may have any sensible physical meaning. Within the ‘kinematical’ ($\kappa_2 \leqslant 0$) interpretation of the real CK spaces, κ_1 is the curvature of spacetime and $\kappa_2 = -1/c^2$ is related to the relativistic constant. A natural query is whether the limits $\eta \to 0, \kappa_1 \to 0$ (and $N \to \infty$) are somehow related to a ‘classical’ limit $\hbar \to 0$ within some sensible ‘quantum’ interpretation of the ‘complex Hermitian’ spaces. This is worth exploring.

There are also other possibly interesting applications of an explicit knowledge of the trigonometry of this family of spaces. We mention four main lines. First, the real spacetime models with zero or constant spacetime curvature (Minkowskian and de Sitter spacetimes) are superseded by a variable curvature pseudo-Riemannian spacetime; this is the essence of the Einstenian interpretation of gravitation. The possibility of a kind of ‘Riemannian’ quantum space of states, whose curvature might not be constant, cannot be precluded *a priori*. To explore and figure out what consequences might follow from this idea, familiarity with their trigonometry is a first-order tool in this aim.

Second, another physical problem where the results we have obtained could apply lies on the use of pseudo-Hilbert spaces with an indefinite Hermitian scalar product (Gupta–Bleuler type). These indefinite quantum spaces of states are those corresponding to $\kappa_2 > 0$; and its Hermitian trigonometry should provide the basic elementary relations in the geometry of these spaces, just as the corresponding real relations are the basic spacetime relations in the de Sitter and anti-de Sitter spacetimes. Third, our basic *‘complex Hermitian’ point loop* and *‘complex Hermitian’ line loop* triangle equations (5.11), (5.12) and (5.13) resemble the Mielnik ‘evolution loops’ [34] (the original identity is also a product of 12 terms). Fourth, the physical relevance of Hermitian symmetric spaces is recurring; see for instance a recent proposal on these spaces in relation to quantum gravity [35]. In all these three problems, a good understanding of the geometry of the Hermitian constant curvature cases might be helpful.

Real hyperbolic trigonometry, deeply involved in manifold classification problems, knot theory etc, is merely a particular case of real CK trigonometry. It is not unreasonable to assume that (some instances of) the generic Hermitian trigonometry may be at least as relevant in the similar ‘complexified’ problems [36]. The intriguing indications for an essentially complex nature of spacetime at some deep level also makes worthy the study of complex spaces in a way as explicit and visual as possible.

Apart from the physical interest of particular results, another potential in the method proposed in [1] and developed in the present paper lies in the possibility of opening an avenue for studying the trigonometry of other symmetric homogeneous spaces, most of whose trigonometries are still unknown. Very few results are known in this area; a general sine theorem is derived in Leuzinger [37] for non-compact spaces, and the trigonometry of the rank-2 spaces $SU(3)$ and $SL(3,\mathbb{C})/SU(3)$ is discussed in [38, 39], heavily relying on the use of the Weyl theorems on invariant theory and characterization of invariants by means of traces of products of matrices.

The trigonometry of the rank-1 'quaternionic hyper-Hermitian' spaces $(Sp(3)/(Sp(1)\otimes Sp(2))$, $Sp(2,1)/(Sp(1)\otimes Sp(2))$, $Sp(2,1)/(Sp(1)\otimes Sp(1,1))$ or $Sp(6,\mathbb{R})/(SO(2,1)\otimes Sp(4,\mathbb{R})))$, which correspond to further CD extensions with a new CD label η_2 and also of the 'octonionic-type' analogues of the Cayley plane, with another CD label η_3 altogether, reduces in some sense to the 'complex' 2D case, since any triangle in these spaces lies on a 'complex' chain. Thus in a sense the study of trigonometry in rank-1 spaces is essentially complete with the spaces of real quadratic and 'complex Hermitian' type. This reduction is not natural however from a purely quaternionic or octonionic viewpoint. Perhaps quaternionic (and also the exceptional octonionic) trigonometry should be understood better. In any case, this kind of approach in a '$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ spirit' fits into the Arnold idea of mathematical trinities; hopefully it may provide a way to the quest [36] for the quaternionic analogue of the Berry phase.

The next natural objective along this line is the study of trigonometry Grassmannians. Only rank two real Grassmannian trigonometry has been approached recently [40, 41, 42]; the higher rank real case, or even the rank two complex case is totally explored. Should the method outlined in this paper be able to produce in a direct form the equations of trigonometry for Grassmannians which are also very relevant spaces in many physical applications, this would be a further step towards a general approach to trigonometry of any symmetric homogeneous space. This goal will require first to group all symmetric homogeneous spaces into CKD families, and then to study trigonometry for each family. Work on this line [3, 43, 44] is in progress.

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